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THE  $6j$ -SYMBOLS  
FOR  
AMBIVALENT GROUPS WITH INTEGER REPRESENTATIONS

by

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ABSTRACT

The  $6j$ -symbols are studied for ambivalent groups with integer representations. The  $6j$ -symbols are defined in terms of  $3j$ -symbols, in analogy with the usual definition for simply reducible groups.

Simple symmetry relations are obtained for the  $6j$ -symbols from the symmetry relations of the  $3j$ -symbols. These  $6j$ -symbols are found to satisfy simple orthogonality relations. Also, relations similar to the back-coupling rule and Biedenharn identity are obtained for groups not multiplicity free. Finally, four group sums are obtained for products of these  $6j$ -symbols. Thus, it is shown that most of the properties of the  $6j$ -symbols for simply reducible groups have their equivalent for groups that are not multiplicity free.



## ACKNOWLEDGEMENT

It is my pleasure to thank my director, Professor W.T. Sharp, who first suggested this problem. Without his competent help this research work could not have been carried out in such a short period of time. The demonstrations of the fourth chapter are mere extensions of the proofs he gave in his Ph.D. thesis for simply reducible groups.

I am very grateful to the National Research Council for their grant during the summer of 1962 during which this work was done.

I also wish to thank my wife Marlene, for helping with some of the tedious calculations and for the typing of this thesis.

Finally, I should like to dedicate this first research effort to the memory of my father.



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## INTRODUCTION

The purpose of this thesis is to study the mathematical properties of certain group invariants. It is believed that these properties may, in the near future, be of importance in applications to theoretical nuclear physics.

The  $6j$ -symbol (actually something equivalent to it) was introduced in the literature by Racah in 1942. The group studied in this famous paper<sup>1</sup> is the rotation group in three dimensions,  $R_3$ . A few years earlier, Wigner had studied the interesting properties of the  $3j$ - and  $6j$ -symbols for simply reducible groups. In 1951, Wigner distributed a few copies of a manuscript<sup>2</sup> on Racah algebra for simply reducible groups. In the foreword to this report he writes: "Fifty of the sixty-two pages of the present note are a verbatim copy of a manuscript that is older than the writer cares to admit." The work that Wigner refers to, seems to have been done in the thirties.

In his Ph.D. thesis, W.T. Sharp<sup>3</sup> developed Racah algebra for a group that is compact, quasi-ambivalent and multiplicity free. He also discussed, there, the possible extension of Racah algebra to the locally compact case.

In 1962, Hamermesh in his book on group theory<sup>4</sup> gives a detailed discussion of the  $3j$ -symbols for the symmetric group  $S_n$ . This section of his book is the content of unpublished work of Crosbie and Hamermesh<sup>5</sup> (1956). The group  $S_n$  for  $n > 4$  is not multiplicity free. However, the symmetric group has only integer representations.



In this thesis, the properties of the  $6j$ -symbols are studied for ambivalent groups with integer representations (of which  $S_n$  is a special case).

When this work originated, the symmetric group only, was to be considered. However, it turned out that the main results could also be obtained for a more general type of group; i.e. the ambivalent groups with integer representations.

In the first chapter a brief review of the theory of groups is presented. This is not meant to be rigorous nor complete. Rather, only the concepts which are to be used later in the thesis are introduced.

Most of the second and third chapters were taken from the book of M. Hamermesh, "Group Theory".<sup>4</sup>

The original part of this work is to be found in the fourth chapter. There, the  $6j$ -symbols are defined and their properties are studied. The results obtained for groups which are not multiplicity free are slightly more complicated than the corresponding formulae for simply reducible groups. The proofs of most of the theorems of this chapter follow closely the proofs of the equivalent theorems (for simply reducible groups) to be found in the Ph.D. thesis of W.T. Sharp<sup>3</sup>.



## CHAPTER 1

### Elements of Group Representation Theory.

This thesis was written with physicists in mind. Many of them are familiar with Racah algebra, but some of the results that we shall need for the last two chapters are not always well-known. This first chapter is not an introduction to group theory. Rather, it contains a series of definitions and useful formulae which are assumed to be known, but are collected here for convenience.

For more details the reader can refer to one of the standard textbooks on group theory, e.g. "Group Theory", by M. Hamermesh.<sup>4</sup>

A group  $G$  is a set of elements  $a, b, c, \dots$ , for which a product of two elements is defined, and which satisfies the following conditions:

- 1- If  $a$  and  $b$  are in  $G$  then so is  $ab$ .
- 2- For all  $a, b, c$ , in  $G$  we have  $(ab)c = a(bc)$ . We write  $(ab)c = a(bc) = abc$ .
- 3- There is an element  $e$  in  $G$ , called the identity, such that for all  $a$  in  $G$ ,  $ae = ea = a$ .
- 4- For each  $a$  in  $G$ , there is an element  $b$  called its inverse such that  $ab = ba = e$ . We denote the element  $b$  as  $a^{-1}$ .

The order of a group is the number of elements in this group. In this thesis, we shall restrict our attention to groups of finite order. A group is said to be commutative or abelian if all the elements commute with one another.

Two elements  $a$  and  $b$  in  $G$  are said to be conjugate to each other if there is an element  $x$  in  $G$ , such that  $x^{-1}ax = b$ .



A class  $A$  is the set of all elements of  $G$  which are conjugate to a given element  $a$ . Clearly,  $a$  belongs to  $A$ . Each element of the group  $G$  belongs to one and only one class. A subgroup  $H$  of a group  $G$  is a group which has all its elements in  $G$ . The group  $G$  itself and the identity element are said to be the two improper subgroups of  $G$ . All others, if any, are said to be proper. We say that two groups  $G$  and  $G'$  are isomorphic if there is between their elements a one-to-one correspondence which preserves the multiplication law, i.e.:

- 1- for each  $a$  in  $G$  there is one and only one  $a'$  in  $G'$  and conversely
- 2- if  $ab = c$ , then  $a'b' = c'$ .

A homomorphism also preserves the multiplication law, but now the correspondence need not be one-to-one any more. Many elements in  $G$  may now correspond to the same element in  $G'$ . In other words,  $G$  is homomorphic to  $G'$  if:

- 1- for each element  $a$  in  $G$  there is one and only one element  $a'$  in  $G'$ , and for each element  $a'$  in  $G'$  there corresponds at least one element of  $G$ ;
- 2-  $ab = c$  implies  $a'b' = c'$ .

A linear vector space  $L$ , is a set of elements  $x, y, \dots$  for which an internal and an external law of composition are defined.  $L$  is an abelian group under the internal law, and the operators for the external law are the field of complex numbers  $a$ . More precisely,  $L$  is a linear vector space if:

- 1- when  $x$  and  $y$  are in  $L$  and  $a$  is a complex number, then  $ax$  and  $x+y = y+x$  are also in  $L$ ;
- 2-  $(a+b)x = ax + bx$ ;



3-  $(ab)x = a(bx)$  ;

4-  $1x = x$  ;

5-  $a(x + y) = ax + ay$  ;

6- there is a null vector 0, such that for all  $x$  in  $L$  :  $x + 0 = x$  .

For example, the set of all  $n$  by  $n$  matrices forms a linear vector space.

A set of vectors  $x_1, x_2, \dots, x_n$  is said to be linearly independent if there is no nontrivial solution to the equation:

$$\sum_{i=1}^n a_i x_i = 0 , \quad (1-1)$$

If there is a nontrivial solution, then these vectors are said to be linearly dependent. We say that the dimension of  $L$  is  $n$  when we can find  $n$  vectors which are linearly independent, while  $n+1$  vectors in  $L$  are always linearly dependent. For example, the vector space formed from the set of all  $n$  by  $n$  matrices has dimension  $n^2$ . In this thesis, we shall restrict our attention to spaces of finite dimension. In the space  $L$ , any set of  $n$  linearly independent vectors is said to form a basis in  $L$ , and the vectors are called the basis vectors. Consider a set of operators defined everywhere in  $L$ . (By an operator defined everywhere in  $L$ , we mean that to each and every vector  $x$  in  $L$ , there corresponds an image vector  $x'$  in  $L$  ). If these operators have the group structure (i.e. they satisfy all the group properties), we say that they are an operator group in  $L$ . Our multiplication convention is such that  $y = RSx$  means:  $y$  is to be obtained by letting  $S$  operate on  $x$  first and then  $R$  operate on  $Sx$ .



Consider an abstract group  $G$ . The elements of  $G$  can possibly be mapped on a group of operators  $D(G)$ , defined in the vector space  $L$ , such that  $D(R)D(S) = D(RS)$ . Then the operator group  $D(G)$  is said to be a representation of the group  $G$  in the representation space  $L$  over the field of complex numbers. If the mapping is an isomorphism, then the representation is said to be faithful. If the dimension of  $L$  is  $n$ , then the representation  $D(G)$  is said to be of degree  $n$ . Once a basis is chosen in  $L$ , the operators  $D(R)$  can be described by their representation matrices.

Let the basis vectors be  $|i\rangle$  and consider the operator  $D(R)$  which represents the element  $R$  of the group  $G$ . The representation matrix  $D(R)$  in this basis is defined by giving its matrix elements:  $D_{ij}(R) = \langle i | D(R) | j \rangle$ . It follows that:  $D(R) | i \rangle = D_{ji}(R) | j \rangle$ .

When we have more than one representation, we distinguish among them by using superscripts: thus  $D_{ij}^{\mu}(R)$  is the  $ij$ 'th element of the matrix representing  $R$  in the irreducible representation  $\mu$ .

With each ordered pair of vectors  $x, y$  in  $L$  we can associate a complex number called the scalar product of  $x$  and  $y$  and denote it  $(x, y)$ . It is required to satisfy the following conditions:

- 1-  $(x, y) = \overline{(y, x)}$  ;
- 2-  $(x, ay) = a(x, y)$  ;
- 3-  $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$  ;
- 4-  $(x, x) \geq 0$  , (equal sign only if  $x = 0$ ).



An operator  $U$  is said to be unitary if  $(Ux, Uy) = (x, y)$  for all  $x$  and  $y$  in the space. In this report, we shall restrict our attention to unitary spaces (i.e. in which a scalar product is defined) and unitary operators. For finite groups there is no loss in generality in doing so.

There may be a subspace  $L'$  of  $L$  such that all vectors of  $L'$  are transformed into vectors of  $L'$  by an operator  $D$ . Then, we say that  $L'$  is invariant under the transformation  $D$ . Suppose that  $L'$  is invariant under all transformations  $D(R)$  where  $R$  is in the group  $G$ , then the transformations  $D'(R)$  induced in  $L'$  will form a representation of the group  $G$ . In this case, we say that the representation  $D(G)$  is reducible, and  $L'$  is said to be an invariant subspace of  $L$  under  $G$ . If there is no invariant subspace in the representation space, we say that the representation based on this representation space is irreducible.

It can be shown that for unitary representations, once an invariant subspace  $L_1$  of  $L$  is found, the vectors which are in  $L$  but not in  $L_1$  also form an invariant subspace of  $L$ . Then,  $L$  can be written as a direct "sum" of  $L_1$  and  $L_2$ . We see then, that the matrix representatives  $D(R)$  must, with suitable choice of basis, break down to two submatrices along the diagonal, i.e. will be of the form:

$$D(R) = \begin{pmatrix} D^1(R) & 0 \\ 0 & D^2(R) \end{pmatrix}, \quad (1-2)$$

where  $D^\mu(R)$  is a square matrix of dimension equal to the dimension of the space  $L_\mu$ . The matrices are of this form because it was agreed to order the basis vectors in  $L$  such that the basis vectors



of  $L_1$  come first.

It must be realized that in the space  $L$ , many different sets of basis vectors can be chosen. Since each set of basis vectors is linearly independent, any one set can be expressed as a linear combination of any other set. The transformation from one set to another is therefore given by a unitary (in the case of unitary spaces) matrix. The group operators are unchanged by a basis transformation. This is clear, since the operators are basis independent. However, their representation matrices are changed by such a transformation. Let the transformation matrix be  $U$ , then the representation matrices in the new basis will be given by:

$$D'(R) = UD(R)U^{-1}. \quad (1-3)$$

The matrices  $D'(G)$  also provide a representation of the group  $G$  mapped on  $D(G)$ . Such sets of representation matrices are said to be equivalent representations. They are simply representation matrices of the group operators expressed in different coordinate systems (different bases).

Once an invariant subspace  $L_1$  has been found in  $L$ , we have seen that  $L_2$  (the space spanned by the vectors in  $L$  and not in  $L_1$ ) is also invariant with respect to the group operators.  $L_2$  is called the complement of  $L_1$  with respect to  $L$ , and we say that  $L$  is the direct sum of  $L_1$  and  $L_2$ . Now,  $L_1$  can, in turn, have an invariant subspace and then,  $L_1$  will be the direct sum of this invariant subspace and its complement with respect to  $L_1$ . Similarly for  $L_2$ . This process can be repeated until none of the subspaces can be broken down any further. Then the total space  $L$  is written as a



direct sum over all these invariant subspaces. We write  $L = \sum_{\mu}^{\oplus} L_{\mu}$ , where  $\oplus$  indicates a direct sum is to be performed. After this is done, the basis in  $L$  can be chosen such that any basis vector is completely in one of the subspaces, and the basis vectors can be ordered such that all of those in  $L_1$  come first, those in  $L_2$  second, and so on. Then the representation matrices of the group operators in this basis are of the form:

$$D(G) = \begin{pmatrix} \boxed{D^1(G)} & & & & \\ & \boxed{D^2(G)} & & & \\ & & \ddots & & \\ & & & \ddots & \\ \circ & & & & \boxed{D^n(G)} \\ & \circ & & & \end{pmatrix}, \quad (1-4)$$

where the  $D^{\mu}(G)$  are the representation matrices for the irreducible representation  $\mu$ .

We often write:

$$D(G) = \sum_{\mu}^{\oplus} a_{\mu} D^{\mu}(G), \quad (1-5)$$

where  $a_{\mu}$  indicates that  $D^{\mu}(G)$  may appear more than once in the decomposition. Equation (1-5) can be interpreted in two ways. The  $D(G)$  and the  $D^{\mu}(G)$  can refer simply to the representations themselves and the sum is a direct sum. The representations are defined up to an equivalence, i.e. only inequivalent representations are considered to be different. On the other hand, in equ. (1-5) we can look at the  $D(G)$  and the  $D^{\mu}(G)$  as the matrices representing the representations. The sum on the right is then illustrated in equ.(1-4). However, in order for this interpretation to be correct, the basis must be a very special one. In most bases, the right-hand side of equ. (1-5) is only equivalent to the left-hand side. In such a case,



we write:

$$D(G) \approx \sum_{\mu}^{\oplus} a_{\mu} D^{\mu}(G) , \quad (1-5a)$$

where the sign  $\approx$  (equivalent to  $\sim$ ) has replaced the equality sign of equ. (1-5).

Let  $D^{\mu}(G)$  and  $D^{\nu}(G)$  be two sets of matrices, each providing a representation of the group  $G$ . The Kronecker product of the two matrices  $D_{ij}^{\mu}(R)$  and  $D_{kl}^{\nu}(R)$  is defined by:

$$D_{ik,jl}^{\mu \times \nu}(R) = [D^{\mu}(R) \times D^{\nu}(R)]_{ik,jl} = D_{ij}^{\mu}(R) D_{kl}^{\nu}(R). \quad (1-6)$$

It is easy to show that the matrices  $D^{\mu \times \nu}(R)$  provide a representation of the group  $G$ . Clearly, if the dimensions of the factor representations are  $n_{\mu}$  and  $n_{\nu}$ , the dimension of the product representation is

$$n_{\mu \times \nu} = n_{\mu} n_{\nu} . \quad (1-7)$$

For the case where  $\mu = \nu$ , the product representation  $D^{\mu \times \nu}$  can be reduced (except for the trivial case  $n_{\mu} = 1$ ) into the direct sum of the symmetric product and the antisymmetric product representations. We write:

$$D^{\mu \times \mu} = [D^{\mu} \times D^{\mu}] \oplus \{D^{\mu} \times D^{\mu}\} ,$$

or

$$D^{\mu \times \mu} = D^{\mu} \textcircled{S} D^{\mu} \oplus D^{\mu} \textcircled{A} D^{\mu} .$$

The representations  $D^{\mu} \textcircled{S} D^{\mu}$  and  $D^{\mu} \textcircled{A} D^{\mu}$  are, in general, reducible. The matrices are given by:



$$\begin{aligned} D^{\mu} \textcircled{S} D^{\mu}(R)_{kl,ij} &= [D^{\mu} \times D^{\mu}]_{kl,ij} \\ &= \frac{1}{2} \left[ D_{ki}^{\mu}(R) D_{lj}^{\mu}(R) + D_{li}^{\mu}(R) D_{kj}^{\mu}(R) \right] \end{aligned} \quad (1-6)'$$

and

$$\begin{aligned} D^{\mu} \textcircled{A} D^{\mu}(R)_{kl,ij} &= \{D^{\mu} \times D^{\mu}\}_{kl,ij} \\ &= \frac{1}{2} \left[ D_{ki}^{\mu}(R) D_{lj}^{\mu}(R) - D_{li}^{\mu}(R) D_{kj}^{\mu}(R) \right]. \end{aligned} \quad (1-6)''$$

Their dimensions are respectively  $\frac{1}{2}n_{\mu}(n_{\mu}+1)$  and  $\frac{1}{2}n_{\mu}(n_{\mu}-1)$ .

The character of a group element  $R$  in a representation  $\mu$  is denoted by  $\chi^{\mu}(R)$ . It is defined as the trace of the matrix  $D^{\mu}(R)$ .

Clearly, the characters are basis independent since :

$$\text{Trace } U(DU^{-1}) = \text{Trace } (DU^{-1})U = \text{Trace } D.$$

For the same reason, the character is a class function since all elements of a class are equivalent to one another. Thus, the characters can also be written as  $\chi_i^{\mu}$ , where  $i$  denotes the class. The concept of character is important in physical applications, mainly because the character is an invariant (with respect to basis transformation).

In this thesis, we shall restrict our attention to unitary representations. For finite groups, there is no loss in generality in doing so (in this case, any representation is equivalent to a unitary representation).

A few important properties will now be given for characters and representation matrices. More details can be found in texts



on the theory of group representations.

1- Orthogonality of the representation matrices:

$$\sum_R D_{il}^{\mu}(R) \overline{D_{jm}^{\nu}(R)} = g/n_{\mu} \delta_{\mu\nu} \delta_{ij} \delta_{lm} , \quad (1-8)$$

where  $g$  is the order of  $G$ ,  $n_{\mu}$  the dimension of  $\mu$ , and only inequivalent representations are considered to be different.

The sum is to be performed over all the group elements. Also:

$$\sum_k D_{ik}^{\mu}(R) \overline{D_{jk}^{\nu}(R)} = \delta_{ij} , \quad (1-8)'$$

which expresses the unitarity of the representation matrices.

2- Orthogonality of characters. If in equ.(1-8) we set  $i = l$  and  $j = m$ , and then sum over  $i$  and  $j$  on each side we obtain:

$$\sum_R \chi^{\mu}(R) \overline{\chi^{\nu}(R)} = g \delta_{\mu\nu} . \quad (1-9)$$

Also:

$$\sum_{\nu} \chi^{\nu}_i \overline{\chi^{\nu}_j} = g/g_j \delta_{ij} , \quad (1-10)$$

where  $g_j$  is the number of elements in the class  $j$  and the sum is over all inequivalent representations. Equ.(1-10) is obtained from equ.(1-9) and the fact that the functions  $\sqrt{g_i/g} \chi^{\nu}_i$



( $\nu$  and  $i$  variables) form a unitary matrix.

3- In the definition of the Kronecker product, i.e. equ.(1-6), if we set  $i = j$  and  $k = l$  and sum over  $i$  and  $k$ , we obtain:

$$\chi^{\mu \times \nu}(R) = \chi^{\mu}(R) \chi^{\nu}(R) \quad (1-11)$$

If  $\mu = \nu$ , the characters of the symmetric and antisymmetric product representations are obtained as above, but using equ.(1-6)' and equ.(1-6)" respectively. The results are:

$$[\chi \times \chi(R)] = \frac{1}{2} [(\chi(R))^2 + \chi(R^2)], \quad (1-11)'$$

$$\{\chi \times \chi(R)\} = \frac{1}{2} [(\chi(R))^2 - \chi(R^2)]. \quad (1-11)''$$

4- In equ.(1-5), taking traces on each side, the character of  $D(R)$  is given by:

$$\chi_i = \sum_{\mu} a_{\mu} \chi_i^{\mu}. \quad (1-12)$$

Multiply equ.(1-12) by  $\chi_i g_i$  and sum over  $i$ . Then, use equ.(1-9) and obtain:

$$a_{\mu} = 1/g \sum_i g_i \chi_i \overline{\chi_i^{\mu}} \quad (1-13)$$



In this thesis, we shall be concerned mainly with real representations, i.e. those for which there is a basis, such that all the matrices are real. The characters of a real representation are real. However, if the characters are real, the representation is not necessarily real (cannot necessarily be brought to real form). If  $\chi(R)$  is real, we can only conclude that each  $D(R)$  is equivalent to  $\overline{D(R)}$ , and there may be no basis for which all the  $D(R)$  are real. Finally, the characters may be complex. Thus, an irreducible representation of a group  $G$  falls in one of three kinds:

- 1- There is a basis for which all the  $D(R)$  are real.
- 2- Each  $D(R)$  is equivalent to  $\overline{D(R)}$ , but the condition for type 1 is not met.
- 3- The characters are not all real.

The representations of the first and second kind are the integer and half-integer representations of Wigner.

Consider two irreducible representations  $\mu$  and  $\nu$  of a group  $G$  and form the Kronecker product  $\mu \times \nu$ . In general, the product representation is reducible. It can therefore be written as a direct sum of the irreducible representations of  $G$ , as in equ.(1-5). If  $C(\mu \nu \sigma)$  is the number of times that  $\sigma$  is contained in  $\mu \times \nu$ , we can write:

$$D^\mu \times D^\nu \equiv D^{\mu \times \nu} = \sum_{\sigma}^{\oplus} C(\mu \nu \sigma) D^{\sigma} \quad . \quad (1-14)$$



Using equations (1-11) and (1-13) we obtain:

$$C(\mu \vee \sigma) = 1/g \sum_R \chi^\mu(R) \chi^\sigma(R) \overline{\chi^\sigma(R)}. \quad (1-15)$$

A group  $G$  is said to be multiplicity free if the  $C(\mu \vee \sigma)$  are never greater than one for any choice of  $\mu \vee \sigma$ . In other words, when the product of any two irreducible representations is decomposed, no irreducible representation appears more than once. A group  $G$  is said to be ambivalent if every element is equivalent to its inverse. That is,  $R$  and  $R^{-1}$  are in the same class for all  $R$  in  $G$ . A group  $G$  is simply reducible if it is ambivalent and multiplicity free. An abelian group is an example of a group which is non-ambivalent (except the trivial case  $C_2 \times C_2 \times \dots \times C_2$ ). For an abelian group, every element is in a class by itself and except for trivial cases, the element and its inverse are different. An example of a group which is not multiplicity free is  $S_5$ : the group of permutations on 5 symbols. This group contains 7 irreducible representations of degrees 1, 4, 5, 6, 5, 4, and 1 respectively. Taking the product of the irreducible representation of degree 6 with itself, the degree of the product representation is  $6 \times 6 = 36$ , by equ.(1-7). Now, even if each irreducible representation is contained once in the decomposition, the sum of the dimensions is only  $1+4+5+6+5+4+1 = 26$ . Clearly, some of the irreducible representations must appear more than once in this decomposition and  $S_5$  is therefore not multiplicity free.



In the theory of group representations a special representation, the regular representation, is used very often to prove some general theorems. It is the faithful representation for which the basis vectors are the elements of the group. Consider the group  $G$  of order  $g$  and let the elements of  $G$  be  $S_1, S_2, \dots, S_g$ . Multiplying each element of the set from the right by  $S_\nu$  permutes the elements of the set because of the group structure of this set. This can be used to define the matrix elements of  $S_\nu$ . Let  $S_i S_\nu = S_{j_i}$  ( $i = 1, \dots, g$ ); then,  $D_{i\ell}(S_\nu) = \delta_{\ell j_i}$ . It is easy to show that the  $D$  matrices, thus defined, form a faithful representation of  $G$ . There is only one nonzero value in each row and column of the  $D$ 's and this value is 1.

There seems to be a misprint in the book of Hamermesh. On page 107, for  $S_\nu S_i = S_{j_i}$  read  $S_i S_\nu = S_{j_i}$ .

In this first chapter, we have introduced the basic concepts which are to be used, later in this thesis. A few important formulae have been collected, there, for convenience. We are now in a position to study the properties of the  $3j$ -symbols. This will be done in the next chapter for ambivalent groups with integer representations.



## CHAPTER 2

### The 3j-Symbols

In this chapter the 3j-symbols are defined and some of their properties are discussed. Most of the content of this chapter can be found in the book of Hamermesh.

We have seen that the representation of a group  $G$  given by the Kronecker product of two irreducible representations of  $G$  can be expressed as a direct sum over the irreducible representations of  $G$ . Let the two factor representations be  $\mu$  and  $\nu$  and assume, for the moment, that  $G$  is simply reducible. Now, choose a basis in each representation space. Denote the vectors in  $\mu$  and  $\nu$  by  $X_i^\mu$  and  $Y_k^\nu$  respectively. We have, then:

$$D^\mu X_j^\mu = D_{ij}^\mu X_i^\mu, \quad (2-1)$$

$$D^\nu Y_l^\nu = D_{kl}^\nu Y_k^\nu, \quad (2-1)'$$

where a sum is implied when two repeated roman indices appear in a term of an equation. This summing convention will be used throughout this thesis. Defining the Kronecker product of two matrices as in chapter 1, and using equ.(2-1) and equ.(2-1)' we obtain:

$$D^{\mu \times \nu} X_j^\mu Y_l^\nu = D_{ik, jl}^{\mu \times \nu} X_i^\mu Y_k^\nu. \quad (2-2)$$

The vectors  $X_j^\mu Y_l^\nu$  (often called the outer product of the vectors  $X_j^\mu$  and  $Y_l^\nu$ ) are, therefore, the basis vectors of the product



representations. These vectors span a space which is, in general, reducible. In such a case, the space which they span is a direct sum of invariant subspaces. In general, not every basis vector is completely in one of the invariant spaces. Then, the Kronecker product of the matrices cannot be written in the form of equ.(1-4). However, a basis transformation can be made by taking linear combinations of the vectors  $X_i^\mu Y_k^\nu$  and the coefficients can be chosen such that, in the new basis, the matrices appear as in equ.(1-4). Let  $S_s^{\lambda \mu \nu}$  be the unitary matrix which performs this transformation. The new basis vectors are given by:

$$Z_s^\lambda = S_s^{\lambda \mu \nu} X_i^\mu Y_k^\nu . \quad (2-3)$$

Now, if the group G is not simply reducible, there may be many sets of functions such as  $Z_s^\lambda$ , i.e. there may be more than one independent set of functions which are linear combinations of  $X_i^\mu Y_k^\nu$  and which transform according to  $D^\lambda$ . To distinguish among the various sets we use the index  $\tau_\lambda$  and write:

$$Z_s^{\lambda \tau_\lambda} = S_s^{\lambda \tau_\lambda \mu \nu} X_i^\mu Y_k^\nu , \quad (2-4)$$

where  $\tau_\lambda$  can take on  $C(\mu \nu \lambda)$  values. As before,  $C(\mu \nu \lambda)$  is the number of times that  $\lambda$  appears in the product of  $\mu$  and  $\nu$ . The vectors transform according to:

$$D(R) Z_s^{\lambda \tau_\lambda} = D_{s's'}^{\lambda \tau_\lambda}(R) Z_{s'}^{\lambda \tau_\lambda} . \quad (2-5)$$



We choose the matrices  $D^{\lambda\tau_\lambda}(R)$  to be the same for all  $\tau_\lambda$  and simply write  $D^\lambda(R)$ . The coefficients  $S_s^{\lambda\tau_\lambda \mu \nu}{}_{i k}$  in equ.(2-4) are to be chosen such that  $Z_s^{\lambda\tau_\lambda}$  transforms according to equ.(2-5). They are called the Clebsch-Gordan (C.G.) coefficients and, apart from a numerical factor, they are the 3j-symbols of Wigner.

The unitarity of the C.G. coefficients means that:

$$S_s^{\lambda\tau_\lambda \mu \nu}{}_{i k} \overline{S_{s'}^{\lambda'\tau_{\lambda'} \mu' \nu'}{}_{i' k'}} = \delta_{\lambda\lambda'} \delta_{\tau_\lambda\tau_{\lambda'}} \delta_{ss'} , \quad (2-6)$$

$$\sum_{\lambda, \tau_\lambda} S_s^{\lambda\tau_\lambda \mu \nu}{}_{i k} \overline{S_s^{\lambda\tau_\lambda \mu' \nu'}{}_{i' k'}} = \delta_{ii'} \delta_{kk'} , \quad (2-7)$$

$$X_i^\mu Y_k^\nu = \sum_{\lambda, \tau_\lambda} Z_s^{\lambda\tau_\lambda} \overline{S_s^{\lambda\tau_\lambda \mu \nu}{}_{i k}} . \quad (2-8)$$

We now show that the C.G. coefficients can be chosen to be real. Consider:

$$\begin{aligned} D(R) Z_s^{\lambda\tau_\lambda} &= D_{ss}^\lambda(R) Z_{s'}^{\lambda\tau_\lambda} \\ &= D_{s's}^\lambda(R) S_{s'}^{\lambda\tau_\lambda \mu \nu}{}_{i k} X_i^\mu Y_k^\nu . \end{aligned}$$

On the other hand:

$$\begin{aligned} D(R) Z_s^{\lambda\tau_\lambda} &= D(R) (X_j^\mu Y_l^\nu) S_s^{\lambda\tau_\lambda \mu \nu}{}_{j l} \\ &= D_{ij}^\lambda(R) D_{kl}^\lambda(R) S_s^{\lambda\tau_\lambda \mu \nu}{}_{j l} X_i^\mu Y_k^\nu . \end{aligned}$$

Since the vectors  $X_i^\mu Y_k^\nu$  form a linearly independent set, we have:



$$D_{ij}^{\mu}(R) D_{kl}^{\nu}(R) S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \ \nu} = D_{s's}^{\lambda}(R) S_{s' \ i \ k}^{\lambda \tau \lambda \ \mu \ \nu} . \quad (2-9)$$

The representations we study here are all integer representations and the D matrices are real. The equations (2-9) are therefore a set of linear equations with real coefficients and the C.G. coefficients can be chosen to be real. The unitarity of the C.G. coefficients becomes an orthogonality relation; i.e. in equations (2-6), (2-7), and (2-8) the bars can be suppressed. Using the orthogonality of the C.G. coefficients and equ.(2-9) we obtain:

$$\begin{aligned} S_{t \ i \ k}^{\lambda' \tau' \lambda' \ \mu \ \nu} D_{ij}^{\mu}(R) D_{kl}^{\nu}(R) S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \ \nu} \\ = D_{s's}^{\lambda}(R) \delta_{\lambda \lambda'} \delta_{\tau \lambda \tau' \lambda'} \delta_{ts'} , \end{aligned} \quad (2-10)$$

$$D_{ij}^{\mu}(R) D_{kl}^{\nu}(R) = \sum_{\lambda, \tau \lambda} S_{s' \ i \ k}^{\lambda \tau \lambda \ \mu \ \nu} D_{s's}^{\lambda}(R) S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \ \nu} . \quad (2-11)$$

Using the orthogonality of the D matrices we have from equ.(2-9):

$$D_{ts}^{\lambda}(R) D_{ij}^{\mu}(R) D_{kl}^{\nu}(R) S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \ \nu} = S_{t \ i \ k}^{\lambda \tau \lambda \ \mu \ \nu} . \quad (2-12)$$

Now equation (2-9) can be transformed by using an orthogonality theorem for the D matrices. We state this theorem without proof. It is the equation number (3-143) of the book of Hamermesh specialized to the case of real representations:

$$\sum_R D_{il}^{\mu}(R) D_{jm}^{\nu}(R) = g/n_{\mu} \delta_{\mu \nu} \delta_{ij} \delta_{lm} . \quad (2-13)$$



(The proof of this theorem is based on Schur's lemmas.) We can use equations (2-13), (2-9), and the orthogonality of the C.G. coefficients to obtain:

$$\sum_R D_{ts}^{\lambda}(R) D_{ij}^{\mu}(R) D_{kl}^{\nu}(R) = g/n_{\lambda} \sum_{\tau_{\lambda}} S_{t \quad i \quad k}^{\lambda \tau_{\lambda} \quad \mu \quad \nu} S_{s \quad j \quad l}^{\lambda \tau_{\lambda} \quad \mu \quad \nu}. \quad (2-14)$$

It follows from equ.(2-14) that:

$$\begin{aligned} 1/n_{\lambda} \sum_{\tau_{\lambda}} (S_{t \quad i \quad k}^{\lambda \tau_{\lambda} \quad \mu \quad \nu})^2 &= 1/n_{\mu} \sum_{\tau_{\mu}} (S_{i \quad t \quad k}^{\mu \tau_{\mu} \quad \lambda \quad \nu})^2 \\ &= 1/n_{\nu} \sum_{\tau_{\nu}} (S_{k \quad i \quad t}^{\nu \tau_{\nu} \quad \mu \quad \lambda})^2 = \text{etc.} \end{aligned} \quad (2-15)$$

To obtain equ.(2-15), set  $t=s$ ,  $i=j$ ,  $k=l$  in equ.(2-14) and permute the factors on the left-hand side. It will prove useful to define the 3j-symbol at this stage. We define the 3j-symbols in terms of the C.G. coefficients:

$$\mathcal{S}_{t \quad i \quad k}^{\lambda \tau_{\lambda} \quad \mu \quad \nu} = \sqrt{\frac{1}{n_{\lambda}}} S_{t \quad i \quad k}^{\lambda \tau_{\lambda} \quad \mu \quad \nu}, \quad (2-16)$$

where  $n_{\lambda}$  is the dimension of  $\lambda$ . The script  $\mathcal{S}$  will be used in this chapter. Later on, we shall use another notation:

$$(\mu \nu \lambda \tau_{\lambda})_{ikt} \equiv \mathcal{S}_{t \quad i \quad k}^{\lambda \tau_{\lambda} \quad \mu \quad \nu} \quad (2-17)$$

and sometimes, we shall even drop the letters  $ikt$ . This new



notation will be most useful in deriving theorems about the 6j-symbols. Appendix A gives a series of useful formulae, each one written in terms of C.G. coefficients and in terms of 3j-symbols (2nd notation). In terms of the 3j-symbols equ.(2-15) becomes:

$$\begin{aligned} \sum_{\tau_\lambda} \left( \begin{matrix} \lambda & \tau_\lambda & \mu & \nu \\ t & i & k \end{matrix} \right)^2 &= \sum_{\tau_\mu} \left( \begin{matrix} \mu & \tau_\mu & \lambda & \nu \\ i & t & k \end{matrix} \right)^2 \\ &= \sum_{\tau_\nu} \left( \begin{matrix} \nu & \tau_\nu & \mu & \lambda \\ k & i & t \end{matrix} \right)^2 = \text{etc.} \end{aligned} \quad (2-15)$$

We have seen that the C.G. coefficients could be taken to be real and we have made this choice. Clearly, from equ.(2-15) the C.G. coefficients are only defined up to a sign. From equ.(2-12), we see that only one choice of sign is given for  $S_{s,j,l}^{\lambda,\tau_\lambda,\mu,\nu}$  ( $\lambda, \tau_\lambda, \mu, \nu$  fixed and  $s, j, l$  variables). We can see in equ.(2-12) how the C.G. coefficients transform under a change of basis in representation space. By an inner basis transformation, we mean one done according to equation (2-12). Notice that only one value of  $\tau_\lambda$  is involved in this transformation, i.e. the linear combinations do not involve vectors from different  $\tau_\lambda$ . We now consider another type of basis transformation which we call outer:

$$U Z_s^{\lambda,\tau_\lambda} = \sum_{\tau'_\lambda} U_{\tau_\lambda \tau'_\lambda} Z_s^{\lambda,\tau'_\lambda}, \quad (2-18)$$

where  $U$  is an orthogonal matrix. An outer transformation keeps the matrices in reduced form since the vectors  $Z_s^{\lambda,\tau_\lambda}$  ( $s$  variable) transform in the same way for any value of  $\tau_\lambda$ . Thus, a linear



combination of these vectors (of the type of equ.(2-18)) also transforms in the same way, i.e. according to  $D^\lambda$ . We see, then, that the basis in which the Kronecker product appears in reduced form is only determined up to an outer transformation. This can be used to impose symmetry relations between the C.G. coefficients. This is done by Hamermesh and is repeated here for convenience.

In view of the orthogonality of the D matrices equation (2-9) can be written as:

$$D_{ij}^M(R) S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \nu} = S_{s' \ i \ k}^{\lambda \tau \lambda \ \mu \nu} D_{s's}^\lambda(R) D_{kl}^\nu(R), \quad (2-19)$$

and using equ.(2-11) we obtain:

$$D_{ij}^M(R) S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \nu} = \sum_{\epsilon, \tau_\epsilon} S_{s' \ i \ k}^{\lambda \tau \lambda \ \mu \nu} S_{t' \ s' \ k}^{\epsilon \tau_\epsilon \ \lambda \ \nu} D_{t't}^\epsilon(R) S_{t \ s \ l}^{\epsilon \tau_\epsilon \ \lambda \ \nu}. \quad (2-20)$$

Using the orthogonality of the C.G. coefficients we obtain:

$$D_{ij}^M(R) S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \nu} S_{t \ s \ l}^{\epsilon \tau_\epsilon \ \lambda \ \nu} = S_{s' \ i \ k}^{\lambda \tau \lambda \ \mu \nu} S_{t' \ s' \ k}^{\epsilon \tau_\epsilon \ \lambda \ \nu} D_{t't}^\epsilon(R). \quad (2-21)$$

If we let:

$$M_{jt} = S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \nu} S_{t \ s \ l}^{\epsilon \tau_\epsilon \ \lambda \ \nu}, \quad (2-22)$$

we see that:



$$D^\mu M = M D^\epsilon . \quad (2-23)$$

From equation (2-23) and Schur's lemmas (see Hamermesh for details), it follows that  $M$  is the null matrix if  $\epsilon \neq \mu$ , and it is a multiple of the unit matrix if  $\epsilon = \mu$ . In equation (2-22), a sum is implied over  $s$  and  $l$ , so that the matrix  $M$  can be written as:

$$\sum_s \begin{matrix} \lambda \tau_\lambda \\ s \end{matrix} \begin{matrix} \mu \nu \\ j \ l \end{matrix} \sum_t \begin{matrix} \epsilon \tau_\epsilon \\ t \end{matrix} \begin{matrix} \lambda \nu \\ s \ l \end{matrix} = \sqrt{n_\lambda / n_\mu} \ m_{\tau_\lambda \tau_\mu}^{(\lambda \mu) \nu} \delta_{\epsilon \mu} \delta_{jt} . \quad (2-24)$$

The factor in the square root is introduced to ensure that  $m$  is an orthogonal matrix. In the superscripts to  $m$ , the letter  $\nu$  is outside the bracket to indicate that the third column of the C.G. coefficients is unchanged on the left. The matrix  $m$  will, in general, depend on  $\lambda$ ,  $\mu$ , and  $\nu$ .

After using the orthogonality of the C.G. coefficients we write equ.(2-24) in terms of 3j-symbols as:

$$\sum_s \begin{matrix} \lambda \tau_\lambda \\ s \end{matrix} \begin{matrix} \mu \nu \\ j \ l \end{matrix} = \sum_{\tau_\mu} m_{\tau_\lambda \tau_\mu}^{(\lambda \mu) \nu} \sum_j \begin{matrix} \mu \tau_\mu \\ j \end{matrix} \begin{matrix} \lambda \nu \\ s \ l \end{matrix} . \quad (2-25)$$

It follows from equ.(2-25) and the reality of the 3j-symbols that the matrix  $m^{(\lambda \mu) \nu}$  is a real orthogonal matrix. It can therefore be used to perform an outer basis transformation.

Another such transformation can be found in a similar way except that, in this case the second column of the C.G. coefficients is left unchanged. Another matrix  $m^{\overleftarrow{\lambda \mu} \nu}$  is then found and can be used to perform another outer transformation. Instead of equations



(2-24) and (2-25) we have:

$$S_{s \ l \ j}^{\lambda \tau_\lambda \ \mu \nu} S_{t \ l \ s}^{\epsilon \tau_\epsilon \ \mu \lambda} = \sqrt{n_\lambda/n_\nu} m_{\tau_\lambda \tau_\nu}^{\lambda \mu \nu} \delta_{\epsilon \nu} \delta_{j t} , \quad (2-24)'$$

$$S_{s \ l \ j}^{\lambda \tau_\lambda \ \mu \nu} = \sum_{\tau_\nu} m_{\tau_\lambda \tau_\nu}^{\lambda \mu \nu} S_{j \ l \ s}^{\nu \tau_\nu \ \mu \lambda} . \quad (2-25)'$$

Note that the third transformation of this type, i.e.  $m^{\lambda(\mu\nu)}$  would not be independent of the two considered here. These outer basis transformations can now be used to impose symmetry relations on the 3j-symbols. We must consider three cases: 1-  $\mu \neq \nu \neq \lambda \neq \mu$ , 2-  $\mu = \nu + \lambda$ , 3-  $\mu = \nu = \lambda$ .

1- Case for which  $\mu \neq \nu \neq \lambda \neq \mu$ .

Suppose that we are given a basis and the 3j-symbols in this basis.

The matrix  $m_{\tau_\lambda \tau_\mu}^{(\lambda\mu)\nu}$  can be used to perform an outer basis transformation

$$m Z_j^{\mu \tau_\lambda} = \sum_{\tau_\mu} m_{\tau_\lambda \tau_\mu}^{(\lambda\mu)\nu} Z_j^{\mu \tau_\mu} . \quad (2-26)$$

In view of equ.(2-4), the 3j-symbols will transform according to:

$$m S_{j \ s \ l}^{\mu \tau_\lambda \ \lambda \ \nu} = \sum_{\tau_\mu} m_{\tau_\lambda \tau_\mu}^{(\lambda\mu)\nu} S_{j \ s \ l}^{\mu \tau_\mu \ \lambda \ \nu} . \quad (2-26)'$$

Equations (2-26) and (2-26)' hold for all j and for all j, s, l respectively. We now have a new basis as a result of this transformation. Using equ.(2-25), we have in this new basis:



$$S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \nu} = S_{j \ s \ l}^{\mu \tau \lambda \ \lambda \ \nu} . \quad (2-27)$$

A second outer basis transformation can be performed with the matrix  $m_{\tau \lambda \tau}^{\lambda \mu \nu}$ . This transformation is independent of the first one because  $\mu \neq \lambda \neq \nu \neq \mu$ . If this transformation is used, then the new 3j-symbols satisfy:

$$S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \nu} = S_{l \ j \ s}^{\nu \tau \lambda \ \mu \ \lambda} . \quad (2-28)$$

Also, it is irrelevant whether we write  $\Psi_j^\mu \Psi_l^\nu$  or  $\Psi_l^\nu \Psi_j^\mu$  since  $\mu \neq \nu$ . Therefore we can choose the 3j-symbols such that:

$$S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \nu} = S_{s \ l \ j}^{\lambda \tau \lambda \ \nu \ \mu} . \quad (2-29)$$

Since  $\mu \neq \lambda \neq \nu$  we can also make:

$$S_{j \ s \ l}^{\mu \tau \lambda \ \lambda \ \nu} = S_{j \ l \ s}^{\mu \tau \lambda \ \nu \ \lambda} , \quad (2-30)$$

$$S_{l \ j \ s}^{\nu \tau \lambda \ \mu \ \lambda} = S_{l \ s \ j}^{\nu \tau \lambda \ \lambda \ \mu} . \quad (2-31)$$

2- Case for which  $\lambda \neq \mu = \nu$ .

In this case the second outer basis transformation is not independent of the first one. We still use the first transformation and obtain:



$$\int_{s \quad j \quad l}^{\lambda \tau \lambda \quad \mu \mu} = \int_{j \quad s \quad l}^{\mu \tau \lambda \quad \lambda \mu} . \quad (2-32)$$

We have seen that the Kronecker product of an irreducible representation with itself is the direct sum of the symmetrized and the antisymmetrized product representations. Each of these may in turn be reducible and written as a direct sum over the irreducible representations of the group. Let  $\Psi_{\ell}^{\mu}$  be the basis functions in  $\mu$ . The basis functions in the Kronecker square are then  $\Psi_{\ell}^{\mu} \Psi_j^{\mu}$ . In the symmetrized square, i.e. in  $[D^{\mu} \times D^{\mu}]$ , the basis functions are symmetric in an interchange of  $\ell$  and  $j$ . In the antisymmetrized product, i.e. in  $\{D^{\mu} \times D^{\mu}\}$ , the basis functions are antisymmetric in an interchange of  $\ell$  and  $j$ . So if we assign the basis functions of  $D^{\mu \times \mu}$ , i.e.  $\Psi_s^{\lambda \tau \lambda}$ , to the symmetrized or antisymmetrized square\*, then:

$$\int_{s \quad j \quad l}^{\lambda \tau \lambda \quad \mu \mu} = \delta_{\tau \lambda} \int_{s \quad l \quad j}^{\lambda \tau \lambda \quad \mu \mu} , \quad (2-33)$$

where  $\delta_{\tau \lambda} = +1$  if the  $\Psi_s^{\lambda \tau \lambda}$  are in the symmetrized product, and  $= -1$  if they are in the antisymmetrized product. On the other hand, since  $\lambda \neq \mu$  we can take:

$$\int_{j \quad s \quad l}^{\mu \tau \lambda \quad \lambda \mu} = \int_{j \quad l \quad s}^{\mu \tau \lambda \quad \mu \lambda} . \quad (2-34)$$

Therefore for case 2, we have the symmetry relations below:

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\* This assignment is only possible with certain choices of basis.



$$\begin{aligned} S_{s j l}^{\lambda \tau \lambda \mu \mu} &= S_{j s l}^{\mu \tau \lambda \lambda \mu} = S_{j l s}^{\mu \tau \lambda \mu \lambda} \\ &= \delta_{\tau \lambda} S_{s l j}^{\lambda \tau \lambda \mu \mu} = \delta_{\tau \lambda} S_{l s j}^{\mu \tau \lambda \lambda \mu} = \delta_{\tau \lambda} S_{l j s}^{\mu \tau \lambda \mu \lambda} . \end{aligned} \quad (2-35)$$

3- Case for which  $\lambda = \mu = \nu$ .

As before we have:

$$S_{s j l}^{\lambda \tau \lambda \lambda \lambda} = \delta_{\tau \lambda} S_{s l j}^{\lambda \tau \lambda \lambda \lambda} . \quad (2-36)$$

For case 3, equ.(2-12) becomes:

$$D_{ts}^{\lambda}(R) D_{ij}^{\lambda}(R) D_{kl}^{\lambda}(R) S_{s j l}^{\lambda \tau \lambda \lambda \lambda} = S_{t i k}^{\lambda \tau \lambda \lambda \lambda} . \quad (2-37)$$

This can be written as (writing  $D_{ts}^{\lambda}(R) D_{ij}^{\lambda}(R) D_{kl}^{\lambda}(R) = A_{tik, sjl}^{\lambda}(R)$ ):

$$[A_{tik, sjl}^{\lambda}(R) - \delta_{ts} \delta_{ij} \delta_{kl}] S_{s j l}^{\lambda \tau \lambda \lambda \lambda} = 0 . \quad (2-38)$$

If  $S_{s j l}^{\lambda \tau \lambda \lambda \lambda}$  is a solution of equ.(2-38), so is  $S_{j s l}^{\lambda \tau \lambda \lambda \lambda}$  since the coefficients are invariant under a simultaneous interchange of  $j$  and  $s$ , and of  $t$  and  $i$ . The sum and difference of these solutions are also solutions, and at most one of them can be taken to be zero. We make:

$$S_{s j l}^{\lambda \tau \lambda \lambda \lambda} = \epsilon_{\tau \lambda} S_{j s l}^{\lambda \tau \lambda \lambda \lambda} , \quad (2-39)$$

where  $\epsilon_{\tau \lambda} = \pm 1$ . From equations (2-36) and (2-39) it follows that  $\delta_{\tau \lambda} = \epsilon_{\tau \lambda}$ . To show this consider:



$$\begin{aligned} \mathcal{S}_{s \ j \ l}^{\lambda \tau \lambda \ \lambda \ \lambda} &= \delta_{\tau \lambda} \mathcal{S}_{s \ l \ j}^{\lambda \tau \lambda \ \lambda \ \lambda} = \delta_{\tau \lambda} \epsilon_{\tau \lambda} \mathcal{S}_{l \ s \ j}^{\lambda \tau \lambda \ \lambda \ \lambda} = \epsilon_{\tau \lambda} \mathcal{S}_{l \ j \ s}^{\lambda \tau \lambda \ \lambda \ \lambda} = \mathcal{S}_{j \ l \ s}^{\lambda \tau \lambda \ \lambda \ \lambda} \\ &= \delta_{\tau \lambda} \mathcal{S}_{j \ s \ l}^{\lambda \tau \lambda \ \lambda \ \lambda} = \delta_{\tau \lambda} \epsilon_{\tau \lambda} \mathcal{S}_{s \ j \ l}^{\lambda \tau \lambda \ \lambda \ \lambda}. \end{aligned} \quad (2-40)$$

Therefore,  $\delta_{\tau \lambda} \epsilon_{\tau \lambda} = +1$  or  $\delta_{\tau \lambda} = \epsilon_{\tau \lambda}$ . Thus, for case 3 we have:

$$\begin{aligned} \mathcal{S}_{s \ j \ l}^{\lambda \tau \lambda \ \lambda \ \lambda} &= \mathcal{S}_{j \ l \ s}^{\lambda \tau \lambda \ \lambda \ \lambda} = \mathcal{S}_{l \ s \ j}^{\lambda \tau \lambda \ \lambda \ \lambda} \\ &= \delta_{\tau \lambda} \mathcal{S}_{s \ l \ j}^{\lambda \tau \lambda \ \lambda \ \lambda} = \delta_{\tau \lambda} \mathcal{S}_{j \ s \ l}^{\lambda \tau \lambda \ \lambda \ \lambda} = \delta_{\tau \lambda} \mathcal{S}_{l \ j \ s}^{\lambda \tau \lambda \ \lambda \ \lambda}. \end{aligned} \quad (2-41)$$

For groups which are not multiplicity free (but have integer representations), the symmetry relations of the 3j-symbols are rather complicated. The symmetry relations of the C.G. coefficients are even more complicated because the square roots of dimensions are involved (see Hamermesh).

A simple rule can be given which summarizes these symmetry relations, i.e. equations (2-27), (2-28), (2-29), (2-30) and (2-31) for case 1, equation (2-35) for case 2, and equation (2-41) for case 3. Consider a 3j-symbol with factor representations A and B, and product representation C, i.e.  $\mathcal{S}_{c \ a \ b}^{C \ \tau \ A \ B} = (ABC\tau)$ .

1- If A, B, and C are different any permutation leaves the 3j-symbol invariant.

2- If two and only two of the three A, B, C are equal:

interchanging B and C gives a  $\delta_{\tau}$ ,

interchanging A and B gives a  $\delta_{\tau}$  if and only if  $A = B$ ,

interchanging A and C gives a  $\delta_{\tau}$  if and only if  $A = C$ .



3- If  $A = B = C$  any transposition (interchanging two) introduces a  $\delta_\tau$ .

These three rules can also be shortened to the following:

- 1- Any transposition of two equal symbols introduces a  $\delta_\tau$ .
- 2- A transposition of two unequal symbols leaves the  $3j$  invariant (except for the case  $B \neq C$ , interchanging B and C when either  $A = B$  or  $A = C$  introduces a  $\delta_\tau$ ).

To simplify the manipulation of formulae containing  $3j$ -symbols we introduce three functions  $f_A$ ,  $f_B$ , and  $f_C$  defined as follows:

$$\begin{aligned} f_A [ABC\tau] &= 1 && \text{if } A, B, \text{ and } C \text{ are all different,} \\ &= \delta_\tau && \text{otherwise,} \end{aligned}$$

$$\begin{aligned} f_B [ABC\tau] &= \delta_\tau && \text{if } A = C, \\ &= 1 && \text{otherwise,} \end{aligned}$$

$$\begin{aligned} f_C [ABC\tau] &= \delta_\tau && \text{if } A = B, \\ &= 1 && \text{otherwise.} \end{aligned}$$

Note that:

$f_A [ABC\tau]$  is symmetric in A, B, and C,

$f_B [ABC\tau]$  is symmetric in A and C,

$f_C [ABC\tau]$  is symmetric in A and B.



In the table below, we show the values taken by  $f_A$ ,  $f_B$ ,  $f_C$  as well as their products for the various possible cases of equality of representations.

Table 2-1

	$A \neq B \neq C \neq A$	$B = C \neq A$	$A = B \neq C$	$A = C \neq B$	$A = B = C$
$f_A$	1	$\delta_\tau$	$\delta_\tau$	$\delta_\tau$	$\delta_\tau$
$f_B$	1	1	1	$\delta_\tau$	$\delta_\tau$
$f_C$	1	1	$\delta_\tau$	1	$\delta_\tau$
$f_A f_B$	1	$\delta_\tau$	$\delta_\tau$	1	1
$f_A f_C$	1	$\delta_\tau$	1	$\delta_\tau$	1
$f_B f_C$	1	1	$\delta_\tau$	$\delta_\tau$	1
$f_A f_B f_C$	1	$\delta_\tau$	1	1	$\delta_\tau$

In considering a product (e.g.  $f_A f_B$ ), the arguments of the functions must be the same. For example, we may have:

$$f_A [\mu \nu \lambda \tau_\lambda] f_B [\nu \mu \lambda \tau_\lambda] .$$

This cannot be looked up in the table. However, in this case using the symmetry of  $f_A$  we obtain:

$$f_A [\mu \nu \lambda \tau_\lambda] = f_A [\nu \mu \lambda \tau_\lambda] ,$$

and the value of  $f_A f_B$  can now be looked up in table (2-1) according



to the equality conditions of  $\mu$ ,  $\nu$  and  $\lambda$ . The values taken by the  $f$ 's depend, of course on the arguments. We often call the three representations in the argument a triad. In general the  $f$ 's depend on the triad and on the order of the representation in the triad. Of course they also depend on the  $\tau$  variables through  $\delta\tau$ .

In manipulating the 3j-symbols, all permutations of columns can be reduced to a series of transpositions. The simple rule to be followed is:

an interchange of A and B introduces  $f_C$ ,

an interchange of A and C introduces  $f_B$ ,

an interchange of B and C introduces  $f_A$ .

Thus

$$(\mu \nu \lambda \tau) = f_C [\nu \mu \lambda \tau] (\nu \mu \lambda \tau) .$$

We now have a simple practical rule for dealing with the symmetry relations of the 3j-symbols. This rule will be useful in deriving the properties of the 6j-symbols to be defined in chapter 4.

We shall now obtain the numerical value of a 3j-symbol for a special case. From equ.(1-15) and the orthogonality of characters we know that  $\lambda \times \mu$  contains the identity representation once (and only once) if and only if  $\lambda = \mu$ . Let the vectors in the factor space be  $\psi_i^\mu$  and  $\psi_j^\mu$  and in the product space  $\psi_i^\mu \psi_j^\mu$ . The only vector in the identity representation  $\Psi_1^0$  is an invariant. The



scalar product  $\sum_i \psi_i^\lambda \psi_i^\mu$  gives an invariant and we have:

$$\Psi_i^0 = \frac{1}{\sqrt{n_\mu}} \sum_i \psi_i^\lambda \psi_i^\mu, \quad (2-42)$$

where we have normalized  $\Psi_i^0$ . Now, we have by the definition of the C.G. coefficients:

$$\Psi_i^0 = S_i^0 \sum_{\lambda \mu} \psi_i^\lambda \psi_i^\mu, \quad (2-43)$$

So that:

$$S_i^0 \sum_{\lambda \mu} \psi_i^\lambda \psi_i^\mu = \sum_{\lambda \mu} S_i^0 \psi_i^\lambda \psi_i^\mu = \frac{1}{\sqrt{n_\mu}} \delta_{\lambda \mu} \delta_{i,j}. \quad (2-44)$$

In this second chapter we have obtained relations between the representation matrices and the 3j-symbols. These relations will be used in proving the theorems of the fourth chapter.



### CHAPTER 3


#### The Symmetric Group $S_n$ .

The symmetric groups  $S_n$  for  $n = 2, 3, 4$  are simply reducible groups. However, for  $n$  greater than 4 they are not multiplicity free. On the other hand, the representations of  $S_n$  can be taken to be real.

We now introduce a few concepts which are useful in dealing with the symmetric group. A partition of  $n$  is a sequence of numbers  $(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_n)$  such that:

$$\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n = n, \quad (3-1)$$

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_n \geq 0. \quad (3-2)$$

It can be shown that the number of partitions of  $n$  is equal to the number of classes (or of inequivalent irreducible representations) of  $S_n$ . Thus, to each partition of  $n$  we associate an irreducible representation and a class of  $S_n$ . To distinguish between the two we denote a class by  $(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$  and a representation by  $(\lambda_1, \lambda_2, \dots, \lambda_n)$ . For the group  $S_5$  we have the representations:  $(5)$ ,  $(4,1)$ ,  $(3,2)$ ,  $(3,1,1)$ ,  $(2,1,1,1)$ , and  $(1,1,1,1,1)$ . We also write the last three as  $(3,1^2)$ ,  $(2,1^3)$ , and  $(1^5)$ . To each partition we associate a Young Diagram, with rows of widths  $\lambda_1, \lambda_2, \dots$ . For example, to  $(3,1^2)$  corresponds the diagram: . We now start placing the digits 1, 2, 3, ... in the squares (one in each square) in such a way that at each stage the number of digits in the first



row is greater than or equal to the number of digits in the second row, etc. When we have placed, in this way, the first  $n$  digits all the squares are filled. We call the resulting figure a standard tableau. For a given diagram, there may be more than one tableau. For example, to  $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$  corresponds  $\begin{smallmatrix} 12 \\ 3 \end{smallmatrix}$  and  $\begin{smallmatrix} 13 \\ 2 \end{smallmatrix}$ . The number of possible tableaux is equal to the degree of the corresponding irreducible representation. We can, therefore, use the tableaux to identify the basis vectors. We can also identify the basis vectors by Yamanouchi symbols (or Y-symbol). A Y-symbol ( $y_n y_{n-1} \dots y_2 y_1$ ) is a sequence of integers such that  $y_i$  is the number of the row in which the number  $i$  appears in the standard tableaux. For example, to  $\begin{smallmatrix} 12 \\ 3 \end{smallmatrix}$  corresponds (3211), to  $\begin{smallmatrix} 134 \\ 2 \end{smallmatrix}$  corresponds (1121), and to  $\begin{smallmatrix} 1256 \\ 34 \end{smallmatrix}$  corresponds (112211). A conjugate representation is one obtained by interchanging rows and columns of the corresponding diagram. Similarly for a conjugate vector. Thus, the conjugate of  $\begin{smallmatrix} 134 \\ 2 \end{smallmatrix}$  is  $\begin{smallmatrix} 12 \\ 3 \\ 4 \end{smallmatrix}$ . A vector will be denoted in this section by  $[\mu i]$ , where  $\mu$  is the irreducible representation and  $i$  the Y-symbol. The conjugate of  $[\mu i]$  will be written  $[\tilde{\mu} \tilde{i}]$ .

For transposition (permutation of only two of the  $n$  objects), it can be shown that the representation matrices have the property:

$$D_{\tilde{i}\tilde{j}}^{\tilde{\lambda}} = D_{ij}^{\lambda} \quad \text{for } i \neq j, \quad (3-3)$$

$$D_{\tilde{i}\tilde{i}}^{\tilde{\lambda}} = - D_{ii}^{\lambda}. \quad (3-4)$$

These matrices also have the property that the non diagonal elements



$D_{ij}^\lambda$  ( $i \neq j$ ) are nonzero only if the Y-symbols for  $i$  and  $j$  are obtained from each other by a transposition of neighboring numbers in the symbol.

By definition, let  $\bigwedge_i^\lambda$  be  $\pm 1$  according to whether the Y-symbol  $i$  is obtained from the Y-symbol with the letters in natural order by an even or odd number of transpositions. The natural order means that  $y_n \ll y_{n-1} \ll \dots \ll y_1$ .

In the preceding chapter, we have obtained symmetry properties of the  $3j$ -symbols for a general group with integer representations. For the group  $S_n$ , more symmetries can be obtained. These symmetries have to do with conjugation of vectors.

We first show that the completely antisymmetric function  $[(1^n), 1]$  is given by:

$$[(1^n), 1] = 1/\sqrt{n!} \sum_i \bigwedge_i^\lambda [\lambda_i] [\tilde{\lambda} \tilde{i}]' . \quad (3-5)$$

We must show that when a transposition is applied to equ.(3-5) a minus sign is obtained, i.e.

$$D(R) [(1^n), 1] = - [(1^n), 1] .$$

Consider:

$$\begin{aligned} D(R) \sum_i \bigwedge_i^\lambda [\lambda_i] [\tilde{\lambda} \tilde{i}]' &= \sum_{j,k} \bigwedge_i^\lambda [\lambda_j] [\tilde{\lambda} \tilde{k}]' D_{ji}^\lambda(R) D_{\tilde{k}\tilde{i}}^{\tilde{\lambda}}(R) \\ &= \sum_{j,k} [\lambda_j] [\tilde{\lambda} \tilde{k}]' \sum_i \bigwedge_i^\lambda D_{ji}^\lambda(R) D_{\tilde{k}\tilde{i}}^{\tilde{\lambda}}(R) . \end{aligned}$$



Since  $R$  is a transposition, we can use equations (3-3) and (3-4) and the expression above becomes:

$$\text{R.H.S.} = \sum_{j,k} [\lambda_j][\tilde{\lambda}_k]' \left\{ \sum_{i \neq k} \Lambda_i^\lambda D_{ji}^\lambda(R) D_{ki}^\lambda(R) - \Lambda_k^\lambda D_{jk}^\lambda(R) D_{kk}^\lambda(R) \right\}.$$

Now, by the property given after equ.(3-4), it follows that:

$$D_{ki}^\lambda(R) = 0 \quad \text{or else} \quad \Lambda_i^\lambda = -\Lambda_k^\lambda.$$

Therefore, we obtain for the expression above:

$$\text{R.H.S.} = \sum_{j,k} [\lambda_j][\tilde{\lambda}_k]' \left\{ -\Lambda_k^\lambda \sum_i D_{ji}^\lambda(R) D_{ki}^\lambda(R) \right\},$$

and using the orthogonality of the  $D$  matrices we obtain:

$$\text{R.H.S.} = - \sum_{j,k} \Lambda_k^\lambda [\lambda_j][\tilde{\lambda}_k]' \delta_{jk} = - \sum_j \Lambda_j^\lambda [\lambda_j][\tilde{\lambda}_j]'$$

This completes the proof of equ.(3-5). It follows from this equation that:

$$S_{1 \quad i \quad \tilde{k}}^{(1'') \quad \lambda \quad \tilde{\lambda}} = S_{1 \quad i \quad \tilde{k}}^{(1'') \quad \lambda \quad \tilde{\lambda}} = 1/\sqrt{n_\lambda} \Lambda_i^\lambda \delta_{ik}, \quad (3-6)$$

It can be shown from a theorem of Frobenius (or by using diagrams) that:

$$\chi_\lambda^\lambda = \chi_\lambda^{\tilde{\lambda}} \chi_{\tilde{\lambda}}^{(1'')} \quad (3-7)$$



From equ.(3-7) and equ.(1-15) it follows that:

$$(\lambda) \times (1^n) = (\tilde{\lambda}), \quad (3-8)$$

Assume that  $\lambda \neq \tilde{\lambda}$ , i.e.  $\lambda$  is not a self-conjugate partition.

We can then write:

$$S_{\tilde{k} i}^{\tilde{\lambda} \lambda (1^n)} = S_{i i}^{(1^n) \lambda \tilde{\lambda}} = 1/\sqrt{n_\lambda} \Lambda_i^\lambda \delta_{ik}, \quad (3-9)$$

where we have used equ.(3-6) and the symmetry properties of the 3j-symbols. It follows from equ.(3-9) that:

$$[\tilde{\lambda} \tilde{i}] = \sum_k S_{\tilde{i} k}^{\tilde{\lambda} \lambda (1^n)} [\lambda k] [(1^n), 1] = \sum_k \Lambda_i^\lambda \delta_{ik} [\lambda k] [(1^n), 1]$$

and

$$[\tilde{\lambda} \tilde{i}] = \Lambda_i^\lambda [\lambda i] [(1^n), 1], \quad (3-10)$$

where  $\lambda \neq \tilde{\lambda}$ . We now consider a representation  $\gamma$  which is contained in  $\tilde{\alpha} \times \tilde{\beta}$ . The basis vectors are given by:

$$[\gamma i] = \sum_{j,k} S_{i j}^{\gamma \tilde{\alpha} \tilde{\beta}} [\tilde{\alpha} j] [\tilde{\beta} k]',$$

and using equ.(3-10)

$$[\gamma i] = \sum_{j,k} S_{i j}^{\gamma \tilde{\alpha} \tilde{\beta}} \Lambda_j^{\tilde{\alpha}} \Lambda_k^{\tilde{\beta}} [\alpha j] [\beta k]' [(1^n), 1] [(1^n), 1]'$$



Now,  $[(1)^n, 1] [(1)^n, 1]'$  is completely symmetric, so that:

$$[\gamma_i] = \sum_{j,k} S_i^{\gamma \tilde{\alpha} \tilde{\beta}} \Lambda_j^{\alpha} \Lambda_k^{\beta} [\alpha_j] [\beta_k]' , \quad (3-11)$$

Using the orthogonality of the C.G. coefficients this becomes:

$$[\alpha_j] [\beta_k]' = \sum_{\gamma,i} \Lambda_j^{\alpha} \Lambda_k^{\beta} S_i^{\gamma \tilde{\alpha} \tilde{\beta}} [\gamma_i] . \quad (3-12)$$

On the other hand:

$$[\alpha_j] [\beta_k]' = \sum_{\gamma,i} S_i^{\gamma \alpha \beta} [\gamma_i] , \quad (3-13)$$

Therefore, comparing coefficients we have the symmetry relation:

$$S_i^{\gamma \tilde{\alpha} \tilde{\beta}} = \Lambda_j^{\alpha} \Lambda_k^{\beta} S_i^{\gamma \alpha \beta}$$

or

$$S_i^{\gamma \tilde{\alpha} \tilde{\beta}} = \Lambda_j^{\alpha} \Lambda_k^{\beta} S_i^{\gamma \alpha \beta} , \quad (3-14)$$

where there is no summation on the repeated indices. Note that we have assumed here that  $\alpha \neq \tilde{\alpha}$ ,  $\beta \neq \tilde{\beta}$ .

Other symmetry relations are obtained in a similar way, by starting with expressions like:

$$[\tilde{\gamma} \tilde{i}] = \sum_{j,k} S_i^{\tilde{\gamma} \tilde{\alpha} \tilde{\beta}} \Lambda_j^{\alpha} \Lambda_k^{\beta} [\tilde{\alpha} \tilde{j}] [\beta_k]' .$$



The results are:

$$S_{\tilde{\gamma} \alpha \tilde{\beta}}^{\tilde{i} j k} = \Lambda_i^{\gamma} \Lambda_k^{\beta} S_{\gamma \alpha \beta}^{\tilde{i} j k}, \quad (3-15)$$

$$S_{\tilde{\gamma} \tilde{\alpha} \beta}^{\tilde{i} \tilde{j} k} = \Lambda_i^{\gamma} \Lambda_j^{\alpha} S_{\gamma \alpha \beta}^{\tilde{i} \tilde{j} k}. \quad (3-16)$$

In equ.(3-15)  $\gamma \neq \tilde{\gamma}$ ,  $\beta \neq \tilde{\beta}$ , and in equ.(3-16)  $\tilde{\gamma} \neq \gamma$ ,  $\tilde{\alpha} \neq \alpha$ .

In the next chapter, we shall see that these special properties of the 3j-symbols for the group  $S_n$  can be used to obtain special symmetry properties of the 6j-symbols for  $S_n$ .



## CHAPTER 4

### The 6j-Symbols.

Introduction. Quantities equivalent to the 6j-symbols were first introduced in the literature by Racah<sup>5</sup>. Since then, Racah algebra for  $R_3$ , the rotation group in three dimensions, has become an important tool of the nuclear physicist. The reason for this, is that spherical symmetry plays an important role in nuclear physics. It would be preposterous to believe that the 6j-symbols for  $S_n$  (the permutation group on n symbols) would become as important as those for  $R_3$ . However, the 6j-symbols for  $S_n$  may be useful for some physical applications. It is also interesting to see how the formalism of Racah algebra carries over for groups that are not multiplicity free. Rather than studying the 6j-symbols for  $S_n$ , we shall consider ambivalent groups with integer representations. Of course,  $S_n$  is an important special case of such groups.

Definition. We shall use the notation of Sharp<sup>3</sup>. The irreducible representations will be denoted by the letter "j" with various subscripts and the vectors by the letter "m" with the corresponding subscripts, e.g.  $j_{12}$  and  $m_{12}$ .

We define the 6j-symbol as follows:

$$\left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} = (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{34} j_{24} j_{23} \tau_1)_{m_{34} m_{24} m_{23}} \\ \times (j_{12} j_{24} j_{14} \tau_3)_{m_{12} m_{24} m_{14}} (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}}, \quad (4-1)$$



where a sum over all possible values of  $m_e$  is implied whenever  $m_e$  appears twice in an expression. This summation convention will be used throughout this chapter. In equ.(4-1), there are six summations implied. Very often, the  $m$ 's will not be written. Then, the summation over  $m_e$  will be implied when  $j_e$  appears twice. Thus equ.(4-1) can be written as:

$$\left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_1 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_2 & \tau_2 \end{array} \right\}$$

$$= (j_{12} j_{13} j_{23} \tau_1) (j_{34} j_{24} j_{14} \tau_1) (j_{12} j_{24} j_{14} \tau_3) (j_{34} j_{13} j_{14} \tau_2). \quad (4-1)'$$

We see that, to each 6j-symbol is associated a set of four triads. The notation, used in the definition, becomes clearer upon considering the following diagram:

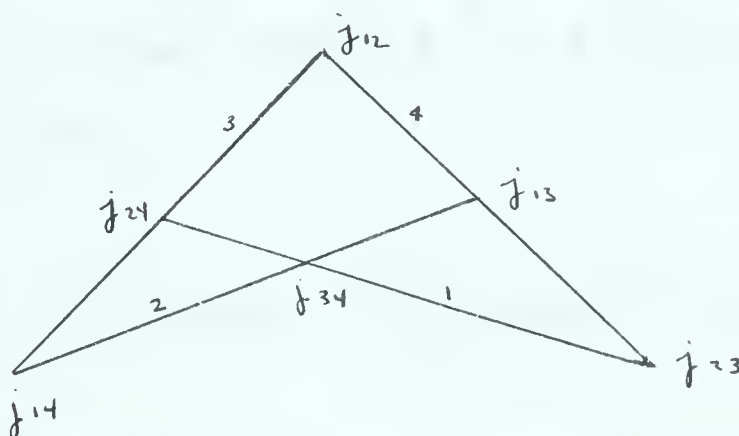


FIGURE 4.1.

Each line represents a triad which is denoted by a single digit; this digit is the only one of 1,2,3 and 4 which does not appear in the subscripts of the 3  $j$ 's forming the triad. The points represent



the irreducible representations. Another helpful diagram is one built on a tetrahedron.

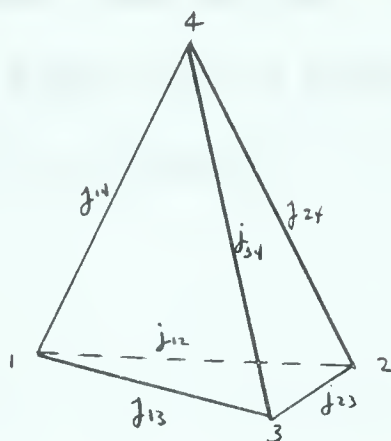


FIGURE 4.2.

Here, each side represents an irreducible representation and each apex represents a triad.

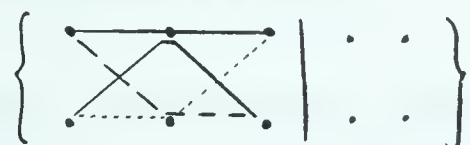
Apart from the six  $j$ 's in the definition, there appear four variables. These four variables may, in general, take more than one set of values. Hence, given the six  $j$ 's, there may be more than one  $6j$ -symbol. The actual number  $N$ , of  $6j$ -symbols for a given set of six  $j$ 's is given by:

$$N = \sum_{\tau_1} \sum_{\tau_2} \sum_{\tau_3} \sum_{\tau_4} 1 = C(j_{12} j_{13} j_{23}) C(j_{12} j_{24} j_{14}) C(j_{34} j_{24} j_{23}) C(j_{34} j_{13} j_{14}),$$

where the same notation as for the definition is used. The sums run over all possible values of the variables and, as usual,  $C(j_1 j_2 j_3)$  is the number of times that  $j_3$  is contained in the Kronecker product of  $j_1$  and  $j_2$ . For groups which are multiplicity free, the value of  $N$ , for any set of six  $j$ 's, is always equal to one. Clearly, to each triad is associated a variable  $\tau$  which can take on values 1, 2, etc. up to and including the value of  $C$  for this triad. Later



on, we shall consider  $6j$ -symbols in which the subscripts to the  $j$ 's will differ from those used above. In each case, the triads are to be formed in the same way as above, i.e. in the diagram below each line refers to a triad:



In words, the triads are to be formed as follows: the first triad is made up of the three  $j$ 's of the first line; to form the other three triads, take one  $j$  from each column such that one appears on the first line and two on the second line in the symbol. It will always be clear, from the subscripts, to which triad a given  $\tau$  belongs moreover, the  $\tau$  variables will appear in the symbol as specified in the definition, e.g. the  $\tau$  belonging to the triad made up of the three  $j$ 's on the first line will appear on the first line and on the left-hand side.

From the definition, the  $6j$ -symbols are seen to be real since the  $3j$ -symbols were taken to be real.

Symmetry Relations of  $6j$ -Symbols. From the definition, a few simple symmetry relations can be obtained. These depend on the symmetry relations satisfied by the  $3j$ -symbols. These symmetry relations could be imposed on the  $3j$ -symbols because there was a certain arbitrariness in the choice of these symbols.

(i) Since the  $3j$ -symbol  $(j_1 j_2 j_3 \tau_{123})$  is taken to be zero when  $j_3$  is not contained in  $j_1 \times j_2$ , it follows that:



$$\left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} = 0 \quad (4-2)$$

unless  $C(j_{12}, j_{13}, j_{23})$ ,  $C(j_{12}, j_{24}, j_{14})$ ,  $C(j_{34}, j_{13}, j_{14})$ , and  $C(j_{34}, j_{24}, j_{23})$  are all nonzero, where the  $C$ 's are defined as for equation (1-14).

(ii) It follows directly from the definition that:

$$\begin{aligned} \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} &= \left\{ \begin{array}{ccc|cc} j_{12} & j_{24} & j_{14} & \tau_3 & \tau_2 \\ j_{34} & j_{13} & j_{23} & \tau_4 & \tau_1 \end{array} \right\} \\ &= \left\{ \begin{array}{ccc|cc} j_{34} & j_{24} & j_{23} & \tau_1 & \tau_4 \\ j_{12} & j_{13} & j_{14} & \tau_2 & \tau_3 \end{array} \right\} \\ &= \left\{ \begin{array}{ccc|cc} j_{34} & j_{13} & j_{14} & \tau_2 & \tau_3 \\ j_{12} & j_{24} & j_{23} & \tau_1 & \tau_4 \end{array} \right\}. \end{aligned} \quad (4-3)$$

In other words, the  $6j$ -symbols are invariant under a permutation of the six  $j$ 's which preserves the triads and the order in which the three  $j$ 's appear in each triad.

(iii) By definition of the  $6j$ -symbol we have:

$$\left\{ \begin{array}{ccc|cc} j_{13} & j_{23} & j_{12} & \tau_4 & \tau_3 \\ j_{24} & j_{14} & j_{34} & \tau_2 & \tau_1 \end{array} \right\} = (j_{13} j_{23} j_{12} \tau_4) (j_{24} j_{14} j_{12} \tau_3) (j_{13} j_{14} j_{34} \tau_2) (j_{24} j_{23} j_{34} \tau_1)$$

and using the symmetry properties of the  $3j$ -symbols the right-hand side becomes:



$$\begin{aligned} \text{R.H.S.} = & f_A [j_{13} j_{23} j_{12} \tau_4] f_A [j_{24} j_{14} j_{12} \tau_3] f_A [j_{13} j_{14} j_{34} \tau_2] f_A [j_{24} j_{23} j_{34} \tau_1] \\ & \times (j_{13} j_{12} j_{23} \tau_4) (j_{24} j_{12} j_{14} \tau_3) (j_{13} j_{34} j_{14} \tau_2) (j_{24} j_{34} j_{23} \tau_1) \end{aligned}$$

and

$$\begin{aligned} \text{R.H.S.} = & f_A [j_{13} j_{23} j_{12} \tau_4] f_A [j_{24} j_{14} j_{12} \tau_3] f_A [j_{13} j_{14} j_{34} \tau_2] f_A [j_{24} j_{23} j_{34} \tau_1] \\ & \times f_C [j_{13} j_{12} j_{23} \tau_4] f_C [j_{24} j_{12} j_{14} \tau_3] f_C [j_{13} j_{34} j_{14} \tau_2] f_C [j_{24} j_{34} j_{23} \tau_1] \\ & \times (j_{12} j_{13} j_{23} \tau_4) (j_{12} j_{24} j_{14} \tau_3) (j_{34} j_{13} j_{14} \tau_2) (j_{34} j_{14} j_{23} \tau_1). \end{aligned}$$

Therefore we obtain:

$$\left\{ \begin{array}{ccc|cc} j_{13} & j_{23} & j_{12} & \tau_4 & \tau_3 \\ j_{24} & j_{14} & j_{34} & \tau_2 & \tau_1 \end{array} \right\} = C \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\}, \quad (4-4)$$

where  $C = f_A [j_{12} j_{13} j_{23} \tau_4] f_C [j_{12} j_{13} j_{23} \tau_4] f_A [j_{12} j_{24} j_{14} \tau_3] f_C [j_{12} j_{24} j_{14} \tau_3]$   
 $\times f_A [j_{34} j_{13} j_{14} \tau_2] f_C [j_{34} j_{13} j_{14} \tau_2] f_A [j_{34} j_{24} j_{23} \tau_1] f_C [j_{34} j_{24} j_{23} \tau_1].$

We see that the  $\delta_j$ -symbols is symmetric (apart from a sign given by C) in the permutation of columns as shown above. The sign is given by C which is a product of eight f functions: one  $f_A$  and one  $f_C$  for each triad. The argument is in the same order as in the  $\delta_j$ -symbol on the right.

(iv) Another symmetry relation is obtained when the first two columns are interchanged. By definition:



$$\left\{ \begin{array}{ccc|cc} j_{13} & j_{12} & j_{23} & \tau_4 & \tau_1 \\ j_{24} & j_{34} & j_{14} & \tau_2 & \tau_3 \end{array} \right\} = (j_{13} j_{12} j_{23} \tau_4) (j_{13} j_{34} j_{14} \tau_2) (j_{24} j_{12} j_{14} \tau_3) (j_{24} j_{34} j_{23} \tau_1) \\ = f_c [j_{13} j_{12} j_{23} \tau_4] f_c [j_{13} j_{34} j_{14} \tau_2] f_c [j_{24} j_{12} j_{14} \tau_3] f_c [j_{24} j_{34} j_{23} \tau_1] \\ \times (j_{12} j_{13} j_{23} \tau_4) (j_{34} j_{13} j_{14} \tau_2) (j_{12} j_{24} j_{14} \tau_3) (j_{34} j_{24} j_{23} \tau_1).$$

Therefore:

$$\left\{ \begin{array}{ccc|cc} j_{13} & j_{12} & j_{23} & \tau_4 & \tau_1 \\ j_{24} & j_{34} & j_{14} & \tau_2 & \tau_3 \end{array} \right\} = C \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\}, \quad (4-5)$$

where :

$$C = f_c [j_{12} j_{13} j_{23} \tau_4] f_c [j_{34} j_{13} j_{14} \tau_2] f_c [j_{12} j_{24} j_{14} \tau_3] f_c [j_{34} j_{24} j_{23} \tau_1].$$

Therefore, interchanging the first two columns of a  $6j$ -symbol leaves it invariant apart from  $C$  where  $C$  is a product of four  $f_c$ . Each of the four triads contributes one  $f_c$  and the argument is in the same order as in the  $6j$ -symbol (either on the right or on the left since  $f_c$  is symmetric in  $A$  and  $B$ ).

(v) For the symmetric group  $S_n$ , we have special symmetry relations. They follow from the definition of the  $6j$ -symbol and equations (3-14), (3-15), and (3-16).

$$\left\{ \begin{array}{ccc|cc} \tilde{j}_{12} & \tilde{j}_{13} & j_{23} & \tau_4 & \tau_1 \\ \tilde{j}_{34} & \tilde{j}_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} = C \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\},$$



where from equation (3-14) we see that

$$C = \bigwedge_{m_{12}}^{j_{12}} \bigwedge_{m_{13}}^{j_{13}} \bigwedge_{m_{12}}^{j_{12}} \bigwedge_{m_{24}}^{j_{24}} \bigwedge_{m_{34}}^{j_{34}} \bigwedge_{m_{13}}^{j_{13}} \bigwedge_{m_{34}}^{j_{34}} \bigwedge_{m_{24}}^{j_{24}}$$

and since each  $\bigwedge$  occurs twice  $C = +1$ .

Therefore:

$$\left\{ \begin{array}{ccc|cc} \tilde{j}_{12} & \tilde{j}_{13} & j_{23} & \tau_4 & \tau_1 \\ \tilde{j}_{34} & \tilde{j}_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} = \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\}.$$

In the same way, it follows that:

$$\left\{ \begin{array}{ccc|cc} \tilde{j}_{12} & j_{13} & \tilde{j}_{23} & \tau_4 & \tau_1 \\ \tilde{j}_{34} & j_{24} & \tilde{j}_{14} & \tau_3 & \tau_2 \end{array} \right\} = \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\},$$

$$\left\{ \begin{array}{ccc|cc} j_{12} & \tilde{j}_{13} & \tilde{j}_{23} & \tau_4 & \tau_1 \\ j_{34} & \tilde{j}_{24} & \tilde{j}_{14} & \tau_3 & \tau_2 \end{array} \right\} = \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\}.$$

In other words, the 6j-symbol is invariant under the conjugation of the j's of two of the three columns. One must make sure here that the j's which are conjugated are not selfconjugate. If some j's are selfconjugate, there may be a minus sign introduced. Apart from this sign, the symmetry properties above are valid, i.e. we should really say that the square of the 6j-symbol is invariant under conjugation of the j's of two of the columns.

Transformations of 6j-Symbols. In chapter 2, we have seen how the 3j-symbols transform under a change of basis. Under an inner basis transformation, they transform according to equ.(2-12), and under



an outer basis transformation they transform according to equ.(2-26)%. From this, we can find how the 6j-symbols transform under these changes of basis.

For an outer basis transformation, we first write an equation similar to equ.(2-26)':

$$U(j_1 j_2 j \tau)_{m_1 m_2 m} = \sum_{\tau'} U_{\tau \tau'}(j_1 j_2 j \tau')_{m_1 m_2 m}, \quad (4-6)$$

It follows from equ.(4-6) and the definition of the 6j-symbols that under an outer basis transformation the 6j-symbols transform according to:

$$U \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} = \sum_{\tau'_1, \tau'_2, \tau'_3, \tau'_4} U_{\tau_1 \tau'_1} U_{\tau_2 \tau'_2} U_{\tau_3 \tau'_3} U_{\tau_4 \tau'_4} \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau'_4 & \tau'_1 \\ j_{34} & j_{24} & j_{14} & \tau'_3 & \tau'_2 \end{array} \right\}. \quad (4-7)$$

The first theorem of this chapter shows how the 6j-symbols transform under an inner basis transformation.

Theorem 4.1: The 6j-symbols are invariant under an inner basis transformation.

Proof: The 3j-symbols transform according to equ.(2-12) under such a transformation. Using equ.(2-12) and the definition of the 6j-symbol we have:



$$\begin{aligned}
 D \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} &= \left[ (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} D_{m_{12} m_{12}}^{j_{12}} D_{m_{13} m_{13}}^{j_{13}} D_{m_{23} m_{23}}^{j_{23}} \right] \\
 &\times \left[ (j_{34} j_{24} j_{23} \tau_1)_{m_{34} m_{24} m_{23}} D_{m_{34} m_{34}}^{j_{34}} D_{m_{24} m_{24}}^{j_{24}} D_{m_{23} m_{23}}^{j_{23}} \right] \\
 &\times \left[ (j_{12} j_{24} j_{14} \tau_3)_{m_{12} m_{24} m_{14}} D_{m_{12} m_{12}}^{j_{12}} D_{m_{24} m_{24}}^{j_{24}} D_{m_{14} m_{14}}^{j_{14}} \right] \\
 &\times \left[ (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}} D_{m_{34} m_{34}}^{j_{34}} D_{m_{13} m_{13}}^{j_{13}} D_{m_{14} m_{14}}^{j_{14}} \right],
 \end{aligned}$$

where equ.(2-12) has been used on each 3j-symbol. On the right-hand side the  $D$  matrices can be grouped in pairs and when the sums are performed over the unprimed  $m$ 's, a Kronecker delta is obtained because of the orthogonality of these matrices. Thus  $D_{m_{12} m_{12}}^{j_{12}} D_{m_{12} m_{12}}^{j_{12}} = \delta_{m_{12} m_{12}}$ . We then get:

$$\begin{aligned}
 D \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} &= (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{34} j_{24} j_{23} \tau_1)_{m_{34} m_{24} m_{23}} \\
 &\times (j_{12} j_{24} j_{14} \tau_3)_{m_{12} m_{24} m_{14}} (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}} \\
 &\times \delta_{m_{12} m_{12}} \delta_{m_{34} m_{34}} \delta_{m_{13} m_{13}} \delta_{m_{24} m_{24}} \delta_{m_{23} m_{23}} \delta_{m_{14} m_{14}}
 \end{aligned}$$

and when the sum is performed over every  $m''$ , we obtain:

$$\begin{aligned}
 D \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} &= (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{34} j_{24} j_{23} \tau_1)_{m_{34} m_{24} m_{23}} \\
 &\times (j_{12} j_{24} j_{14} \tau_3)_{m_{12} m_{24} m_{14}} (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}}
 \end{aligned}$$

By definition, the right-hand side is just the untransformed 6j-symbol. Therefore the 6j-symbols are invariant under an inner basis transformation. Q.E.D.



Associativity of the Kronecker Product: The two theorems of this section are a direct consequence of the associativity of Kronecker multiplication. In the language of the rotation group, these theorems are interpreted as connections between different coupling schemes.

Theorem 4.2:

$$\begin{aligned}
 & (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{34} j_{24} j_{23} \tau_1)_{m_{34} m_{24} m_{23}} \\
 &= \sum_{j_{14}, \tau_2, \tau_3} n_{14} (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}} (j_{12} j_{24} j_{14} \tau_3)_{m_{12} m_{24} m_{14}} \\
 & \quad \times \left\{ \begin{array}{c|cc} j_{12} & j_{13} & j_{23} \\ \hline j_{34} & j_{24} & j_{14} \end{array} \left| \begin{array}{cc} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{array} \right. \right\} \quad (4-8)
 \end{aligned}$$

where  $n_{14}$  is the dimension of  $j_{14}$ .

Proof: This theorem is a result of the associativity of the Kronecker product  $j_{12} \times j_{13} \times j_{34}$ . Clearly:

$$(D_{m_{12} m_{12}'}^{j_{12}} D_{m_{13} m_{13}'}^{j_{13}}) D_{m_{34} m_{34}'}^{j_{34}} = D_{m_{12} m_{12}'}^{j_{12}} (D_{m_{13} m_{13}'}^{j_{13}} D_{m_{34} m_{34}'}^{j_{34}}). \quad (4-9)$$

We shall need equ.(2-11). In our notation, this is written:

$$D_{m_1 m_1'}^{j_1} D_{m_2 m_2'}^{j_2} = \sum_{J, \tau_J} n_J (j_1 j_2 J \tau_J)_{m_1 m_2 M} (j_1 j_2 J \tau_J)_{m_1' m_2' M'} D_{M M'}^J. \quad (4-10)$$

Consider the left-hand side of equ.(4-9). Using equ.(4-10) on the term inside the bracket we have:



$$L.H.S. = \sum_{j_{23}, \tau_4} n_{23} (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{12} j_{13} j_{23} \tau_4)_{m'_{12} m'_{13} m'_{23}} D_{m_{23} m'_{23}}^{j_{23}} D_{m_{34} m'_{34}}^{j_{34}}$$

and using equ.(4-10) again we get:

$$L.H.S. = \sum_{\substack{j_{23}, \tau_4, \\ j_{24}, \tau_1}} n_{23} n_{24} (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{12} j_{13} j_{23} \tau_4)_{m'_{12} m'_{13} m'_{23}} \\ \times (j_{34} j_{23} j_{24} \tau_1)_{m_{34} m_{23} m_{24}} (j_{34} j_{23} j_{24} \tau_1)_{m'_{34} m'_{23} m'_{24}} D_{m_{24} m'_{24}}^{j_{24}} .$$

Using equ.(4-10) on the right-hand side of equ.(4-9) we have:

$$R.H.S. = \sum_{j_{14}, \tau_2} n_{14} (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}} (j_{34} j_{13} j_{14} \tau_2)_{m'_{34} m'_{13} m'_{14}} D_{m_{14} m'_{14}}^{j_{14}} D_{m_{12} m'_{12}}^{j_{12}}$$

and using equ.(4-10) again we obtain:

$$R.H.S. = \sum_{\substack{j_{14}, \tau_2, \\ j_{24}, \tau_3}} n_{14} n_{24} (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}} (j_{34} j_{13} j_{14} \tau_2)_{m'_{34} m'_{13} m'_{14}} \\ \times (j_{12} j_{14} j_{24} \tau_3)_{m_{12} m_{14} m_{24}} (j_{12} j_{14} j_{24} \tau_3)_{m'_{12} m'_{14} m'_{24}} D_{m_{24} m'_{24}}^{j_{24}} .$$

We can equate the coefficients of the two expressions obtained:



$$\begin{aligned}
 & \sum_{j_{23}, \tau_1, \tau_4} n_{23} (j_{12} j_{13} j_{23} \tau_4) m_{12} m_{13} m_{23} (j_{12} j_{13} j_{23} \tau_4) m_{12}' m_{13}' m_{23}' \\
 & \quad \times (j_{34} j_{23} j_{24} \tau_1) m_{34} m_{23} m_{24} (j_{34} j_{23} j_{24} \tau_1) m_{34}' m_{23}' m_{24}' \\
 & = \sum_{j_{14}, \tau_1, \tau_3} n_{14} (j_{34} j_{13} j_{14} \tau_2) m_{34} m_{13} m_{14} (j_{34} j_{13} j_{14} \tau_2) m_{34}' m_{13}' m_{14}' \\
 & \quad \times (j_{12} j_{14} j_{24} \tau_3) m_{12} m_{14} m_{24} (j_{12} j_{14} j_{24} \tau_3) m_{12}' m_{14}' m_{24}' .
 \end{aligned}$$

We can use the symmetry relations of the 3j-symbols to obtain:

$$\begin{aligned}
 & \sum_{j_{23}, \tau_1, \tau_4} n_{23} (j_{12} j_{13} j_{23} \tau_4) m_{12} m_{13} m_{23} (j_{12} j_{13} j_{23} \tau_4) m_{12}' m_{13}' m_{23}' (j_{34} j_{24} j_{23} \tau_1) m_{34} m_{24} m_{23} \\
 & \quad \times (j_{34} j_{24} j_{23} \tau_1) m_{34}' m_{24}' m_{23}' \quad f_A [j_{34} j_{24} j_{23} \tau_1] f_A [j_{34} j_{24} j_{23} \tau_1] \\
 & = \sum_{j_{14}, \tau_1, \tau_3} n_{14} (j_{34} j_{13} j_{14} \tau_2) m_{34} m_{13} m_{14} (j_{34} j_{13} j_{14} \tau_2) m_{34}' m_{13}' m_{14}' (j_{12} j_{24} j_{14} \tau_3) m_{12} m_{24} m_{14} \\
 & \quad \times (j_{12} j_{24} j_{14} \tau_3) m_{12}' m_{24}' m_{14}' \quad f_A [j_{12} j_{24} j_{14} \tau_3] f_A [j_{12} j_{24} j_{14} \tau_3], \quad (4-11)
 \end{aligned}$$

Note that  $f_A^2 = 1$ . Now, multiply each side of equ.(4-11) by

$$(j_{12} j_{13} \hat{j}_{23} \hat{\tau}_4) m_{12}' m_{13}' \hat{m}_{23} (j_{34} j_{24} \hat{j}_{23} \hat{\tau}_1) m_{34}' m_{24}' \hat{m}_{23}$$

and carry out the indicated summations. The left-hand side becomes:



$$\begin{aligned} \text{L.H.S.} = & \sum_{j_{23}, \tau_1, \tau_4} n_{23} (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{34} j_{24} j_{23} \tau_1)_{m_{34} m_{24} m_{23}} \\ & \times (j_{12} j_{13} j_{23} \tau_4)_{m'_{12} m'_{13} m'_{23}} (j_{12} j_{13} \hat{j}_{23} \hat{\tau}_4)_{m'_{12} m'_{13} \hat{m}_{23}} \\ & \times (j_{34} j_{24} j_{23} \tau_1)_{m'_{34} m'_{24} m'_{23}} (j_{34} j_{24} \hat{j}_{23} \hat{\tau}_1)_{m'_{34} m'_{24} \hat{m}_{23}} \end{aligned}$$

and using the orthogonality of the 3j-symbols we have:

$$\begin{aligned} \text{L.H.S.} = & \sum_{j_{23}, \tau_1, \tau_4} n_{23} (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{34} j_{24} j_{23} \tau_1)_{m_{34} m_{24} m_{23}} \\ & \times (1/n_{\hat{23}})^2 \delta_{m'_{23} \hat{m}_{23}} \delta_{j_{23} \hat{j}_{23}} \delta_{\tau_4 \hat{\tau}_4} \delta_{m'_{23} \hat{m}_{23}} \delta_{j_{23} \hat{j}_{23}} \delta_{\tau_1 \hat{\tau}_1} , \end{aligned}$$

and

$$\text{L.H.S.} = (j_{12} j_{13} \hat{j}_{23} \hat{\tau}_4)_{m_{12} m_{13} \hat{m}_{23}} (j_{34} j_{24} \hat{j}_{23} \hat{\tau}_1)_{m_{34} m_{24} \hat{m}_{23}} ,$$

where we have used the fact that:

$$\sum_{m'_{23}, \hat{m}_{23}} \delta_{m'_{23} \hat{m}_{23}} = \sum_{\hat{m}_{23}} 1 = n_{\hat{23}} ,$$

and where  $n_{\hat{23}}$  is the degree of  $\hat{j}_{23}$ . On the other hand, the right-hand side of equ.(4-11) becomes:

$$\begin{aligned} \text{R.H.S.} = & \sum_{j_{14}, \tau_2, \tau_3} n_{14} (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}} (j_{12} j_{24} j_{14} \tau_3)_{m_{12} m_{24} m_{14}} \\ & \times (j_{34} j_{13} j_{14} \tau_2)_{m'_{34} m'_{13} m'_{14}} (j_{12} j_{24} j_{14} \tau_3)_{m'_{12} m'_{24} m'_{14}} \\ & \times (j_{12} j_{13} \hat{j}_{23} \hat{\tau}_4)_{m'_{12} m'_{13} \hat{m}_{23}} (j_{34} j_{24} \hat{j}_{23} \hat{\tau}_1)_{m'_{34} m'_{24} \hat{m}_{23}} . \end{aligned}$$



By the definition of the 6j-symbol, this can be written as:

$$\begin{aligned} \text{R.H.S.} = \sum_{j_{14}, \tau_2, \tau_3} n_{14} (j_{34} j_{13} j_{14} \tau_2) m_{34} m_{13} m_{14} (j_{12} j_{24} j_{14} \tau_3) m_{12} m_{24} m_{14} \\ \times \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & \hat{j}_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} . \end{aligned}$$

Therefore, we obtain:

$$(j_{12} j_{13} j_{23} \tau_4) (j_{34} j_{24} j_{23} \tau_1)$$

$$= \sum_{j_{14}, \tau_2, \tau_3} n_{14} (j_{34} j_{13} j_{14} \tau_2) (j_{12} j_{24} j_{14} \tau_3) \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & \hat{j}_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} .$$

Q.E.D.

Theorem 4.2a: The 6j-symbols satisfy the following relation:

$$(j_{34} j_{13} j_{14} \tau_2) (j_{12} j_{24} j_{14} \tau_3)$$

$$= \sum_{j_{14}, \tau_1, \tau_4} n_{23} (j_{12} j_{13} j_{23} \tau_4) (j_{34} j_{24} j_{23} \tau_1) \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & \hat{j}_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} . \quad (4-8)'$$

Proof: The proof is similar to the above. Multiply equation (4-11)

by  $(j_{34} j_{13} \hat{j}_{14} \hat{\tau}_2) m'_{34} m'_{13} \hat{m}_{14}$   $(j_{12} j_{24} \hat{j}_{14} \hat{\tau}_3) m'_{12} m'_{24} \hat{m}_{14}$  and carry out the indicated summations. The right-hand side becomes:

$$\begin{aligned} \text{R.H.S.} = \sum_{j_{14}, \tau_2, \tau_3} n_{14} (j_{34} j_{13} j_{14} \tau_2) m_{34} m_{13} m_{14} (j_{12} j_{24} j_{14} \tau_3) m_{12} m_{24} m_{14} \\ \times (j_{34} j_{13} j_{14} \tau_2) m'_{34} m'_{13} m_{14} (j_{34} j_{13} \hat{j}_{14} \hat{\tau}_2) m'_{34} m'_{13} \hat{m}_{14} \\ \times (j_{12} j_{24} j_{14} \tau_3) m'_{12} m'_{24} m_{14} (j_{12} j_{24} \hat{j}_{14} \hat{\tau}_3) m'_{12} m'_{24} \hat{m}_{14} , \end{aligned}$$



and using the orthogonality of the 3j-symbols we have:

$$\begin{aligned} \text{R.H.S.} = \sum_{j_{14}, \tau_2, \tau_3} n_{14} (j_{34} j_{13} j_{14} \tau_2) (j_{12} j_{24} j_{14} \tau_3) \\ \times (1/n_{14})^2 \delta_{j_{14} \hat{j}_{14}} \delta_{j_{14} \hat{j}_{14}} \delta_{m_{14} \hat{m}_{14}} \delta_{m_{14} \hat{m}_{14}} \delta_{\tau_2 \hat{\tau}_2} \delta_{\tau_3 \hat{\tau}_3} \end{aligned}$$

and

$$\text{R.H.S.} = (j_{34} j_{13} \hat{j}_{14} \hat{\tau}_2) (j_{12} j_{24} \hat{j}_{14} \hat{\tau}_3).$$

On the other hand, the left-hand side of equ.(4-11) becomes:

$$\begin{aligned} \text{L.H.S.} = \sum_{j_{23}, \tau_1, \tau_4} n_{23} (j_{12} j_{13} j_{23} \tau_4) m_{12} m_{13} m_{23} (j_{34} j_{24} j_{23} \tau_1) m_{34} m_{24} m_{23} \\ \times (j_{12} j_{13} j_{23} \tau_4) m_{12}' m_{13}' m_{23}' (j_{34} j_{24} j_{23} \tau_1) m_{34}' m_{24}' m_{23}' \\ \times (j_{34} j_{13} \hat{j}_{14} \hat{\tau}_2) m_{34}' m_{13}' \hat{m}_{14} (j_{12} j_{24} \hat{j}_{14} \hat{\tau}_3) m_{12}' m_{24}' \hat{m}_{14}. \end{aligned}$$

By the definition of the 6j-symbol, this can be written as:

$$\text{L.H.S.} = \sum_{j_{23}, \tau_1, \tau_4} n_{23} (j_{12} j_{13} j_{23} \tau_4) (j_{34} j_{24} j_{23} \tau_1) \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & \hat{j}_{14} & \hat{\tau}_3 & \hat{\tau}_2 \end{array} \right\}.$$

Therefore, we obtain:

$$\begin{aligned} (j_{34} j_{13} j_{14} \tau_2) (j_{12} j_{24} j_{14} \tau_3) \\ = \sum_{j_{23}, \tau_1, \tau_4} n_{23} (j_{12} j_{13} j_{23} \tau_4) (j_{34} j_{24} j_{23} \tau_1) \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\}. \end{aligned}$$

Q.E.D.



Orthogonality of the 6j-Symbols.

Theorem 4.3: The 6j-symbols satisfy the orthogonality relations:

$$\sum_{j_{14}, \tau_1, \tau_3} n_{23} n_{14} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ & j_{34} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & \hat{j}_{23} \\ & j_{34} & j_{14} \end{matrix} \middle| \begin{matrix} \hat{\tau}_4 & \hat{\tau}_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} = \delta_{j_{23} \hat{j}_{23}} \delta_{\tau_1 \hat{\tau}_1} \delta_{\tau_4 \hat{\tau}_4}, \quad (4-12)$$

$$\sum_{j_{23}, \tau_1, \tau_4} n_{23} n_{14} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ & j_{34} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ & j_{34} & \hat{j}_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \hat{\tau}_3 & \hat{\tau}_2 \end{matrix} \right\} = \delta_{j_{14} \hat{j}_{14}} \delta_{\tau_2 \hat{\tau}_2} \delta_{\tau_3 \hat{\tau}_3}. \quad (4-12)'$$

Proof: Using theorem 4.2 we have:

$$\begin{aligned} & (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{34} j_{24} j_{23} \tau_1)_{m_{34} m_{24} m_{23}} \\ & \times (j_{12} j_{13} \hat{j}_{23} \hat{\tau}_4)_{m_{12} m_{13} \hat{m}_{23}} (j_{34} j_{24} \hat{j}_{23} \hat{\tau}_1)_{m_{34} m_{24} \hat{m}_{23}} \\ & = \sum_{\substack{j_{14}, \tau_1, \tau_3, \\ \hat{j}_{14}, \hat{\tau}_1, \hat{\tau}_3}} n_{14} n_{\hat{14}} (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}} (j_{12} j_{24} j_{14} \tau_3)_{m_{12} m_{24} m_{14}} \\ & \quad \times (j_{34} j_{13} \hat{j}_{14} \hat{\tau}_2)_{m_{34} m_{13} \hat{m}_{14}} (j_{12} j_{24} \hat{j}_{14} \hat{\tau}_3)_{m_{12} m_{24} \hat{m}_{14}} \\ & \quad \times \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ & j_{34} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & \hat{j}_{23} \\ & j_{34} & \hat{j}_{14} \end{matrix} \middle| \begin{matrix} \hat{\tau}_4 & \hat{\tau}_1 \\ \hat{\tau}_3 & \hat{\tau}_2 \end{matrix} \right\}. \end{aligned}$$

The orthogonality of the 3j-symbols can be used on each side and:

$$\text{L.H.S.} = (1/n_{23})^2 \delta_{j_{23} \hat{j}_{23}} \delta_{\tau_4 \hat{\tau}_4} \delta_{m_{13} \hat{m}_{13}} \delta_{m_{23} \hat{m}_{23}} \delta_{\tau_1 \hat{\tau}_1},$$

and

$$\text{L.H.S.} = 1/n_{23} \delta_{j_{23} \hat{j}_{23}} \delta_{\tau_4 \hat{\tau}_4} \delta_{\tau_1 \hat{\tau}_1}.$$



On the other hand:

$$\begin{aligned}
 \text{R.H.S.} &= \sum_{\substack{j_{14}, \tau_2, \tau_3 \\ \hat{j}_{14}, \hat{\tau}_2, \hat{\tau}_3}} \delta_{j_{14} \hat{j}_{14}} \delta_{\tau_2 \hat{\tau}_2} \delta_{\tau_3 \hat{\tau}_3} \delta_{m_{14} \hat{m}_{14}} \delta_{m_{14} \hat{m}_{14}} \\
 &\quad \times \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & \hat{j}_{23} \\ j_{34} & j_{24} & \hat{j}_{14} \end{matrix} \middle| \begin{matrix} \hat{\tau}_4 & \hat{\tau}_1 \\ \hat{\tau}_3 & \hat{\tau}_2 \end{matrix} \right\} \\
 &= \sum_{j_{14}, \tau_2, \tau_3} n_{14} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & \hat{j}_{23} \\ j_{34} & j_{24} & \hat{j}_{14} \end{matrix} \middle| \begin{matrix} \hat{\tau}_4 & \hat{\tau}_1 \\ \hat{\tau}_3 & \hat{\tau}_2 \end{matrix} \right\}
 \end{aligned}$$

and we obtain the required result:

$$\sum_{j_{14}, \tau_2, \tau_3} n_{14} n_{23} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & \hat{j}_{23} \\ j_{34} & j_{24} & \hat{j}_{14} \end{matrix} \middle| \begin{matrix} \hat{\tau}_4 & \hat{\tau}_1 \\ \hat{\tau}_3 & \hat{\tau}_2 \end{matrix} \right\} = \delta_{j_{23} \hat{j}_{23}} \delta_{\tau_4 \hat{\tau}_4} \delta_{\tau_1 \hat{\tau}_1} .$$

This is equ.(4-12). We now use theorem 4.2a on the following sum of products:

$$\begin{aligned}
 & (j_{34} j_{13} j_{14} \tau_2) m_{34} m_{13} m_{14} (j_{12} j_{24} j_{14} \tau_3) m_{12} m_{24} m_{14} \\
 & \quad \times (j_{34} j_{13} \hat{j}_{14} \hat{\tau}_2) m_{34} m_{13} \hat{m}_{14} (j_{12} j_{24} \hat{j}_{14} \hat{\tau}_3) m_{12} m_{24} \hat{m}_{14} \\
 &= \sum_{\substack{j_{23}, \tau_1, \tau_4 \\ \hat{j}_{23}, \hat{\tau}_1, \hat{\tau}_4}} n_{23} n_{\hat{23}} (j_{12} j_{13} j_{23} \tau_4) m_{12} m_{13} m_{23} (j_{34} j_{24} j_{23} \tau_1) m_{34} m_{24} m_{23} \\
 & \quad \times (j_{12} j_{13} \hat{j}_{23} \hat{\tau}_4) m_{12} m_{13} \hat{m}_{23} (j_{34} j_{24} \hat{j}_{23} \hat{\tau}_1) m_{34} m_{24} \hat{m}_{23} \\
 & \quad \times \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & \hat{j}_{23} \\ j_{34} & j_{24} & \hat{j}_{14} \end{matrix} \middle| \begin{matrix} \hat{\tau}_4 & \hat{\tau}_1 \\ \hat{\tau}_3 & \hat{\tau}_2 \end{matrix} \right\} .
 \end{aligned}$$

Using the orthogonality of the 3j-symbols on each side we obtain:



$$L.H.S. = (1/n_{14})^2 \delta_{j_{14} \hat{j}_{14}} \delta_{\tau_2 \hat{\tau}_2} \delta_{\tau_3 \hat{\tau}_3} \delta_{m_{14} \hat{m}_{14}} \delta_{m_{14} \hat{m}_{14}}$$

$$L.H.S. = 1/n_{14} \delta_{j_{14} \hat{j}_{14}} \delta_{\tau_2 \hat{\tau}_2} \delta_{\tau_3 \hat{\tau}_3}.$$

On the other hand:

$$\begin{aligned} R.H.S. &= \sum_{\substack{j_{23}, \tau_1, \tau_4, \\ \hat{j}_{23}, \hat{\tau}_1, \hat{\tau}_4}} \delta_{j_{23} \hat{j}_{23}} \delta_{\tau_1 \hat{\tau}_1} \delta_{\tau_4 \hat{\tau}_4} \delta_{m_{23} \hat{m}_{23}} \delta_{m_{23} \hat{m}_{23}} \\ &\quad \times \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & \hat{j}_{23} \\ j_{34} & j_{24} & \hat{j}_{14} \end{matrix} \middle| \begin{matrix} \hat{\tau}_4 & \hat{\tau}_1 \\ \hat{\tau}_3 & \hat{\tau}_2 \end{matrix} \right\} \\ &= \sum_{j_{23}, \tau_1, \tau_4} n_{23} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & \hat{j}_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \hat{\tau}_3 & \hat{\tau}_2 \end{matrix} \right\} \end{aligned}$$

and we obtain the required result:

$$\sum_{j_{23}, \tau_1, \tau_4} n_{23} n_{14} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & \hat{j}_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \hat{\tau}_3 & \hat{\tau}_2 \end{matrix} \right\} = \delta_{j_{14} \hat{j}_{14}} \delta_{\tau_2 \hat{\tau}_2} \delta_{\tau_3 \hat{\tau}_3}. \quad (4-12)$$

Q.E.D.

Back-Coupling Rule: We now prove a theorem which is analogous to the back-coupling rule of the rotation group. The significance of this is explained by Fano and Racah in their book: "Irreducible Tensorial Sets."<sup>6</sup>

Theorem 4.4. The 6j-symbols satisfy the following relation:

$$\left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} = \sum_{j, \tau, \tau'} C n_j \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{24} & j_{34} & j \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau & \tau' \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{24} & j_{14} \\ j_{13} & j_{34} & j \end{matrix} \middle| \begin{matrix} \tau_3 & \tau_2 \\ \tau & \tau' \end{matrix} \right\}, \quad (4-13)$$



where  $C = f_c [j_{24} j_{34} j_{23} \tau_1] f_c [j_{13} j_{34} j_{14} \tau_2] f_c [j_{24} j_{13} j \tau]$ .

Proof: By definition of the 6j-symbol, the left-hand side is:

$$\begin{aligned} \text{L.H.S.} = & (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{34} j_{24} j_{23} \tau_1)_{m_{34} m_{24} m_{23}} \\ & \times (j_{12} j_{24} j_{14} \tau_3)_{m_{12} m_{24} m_{14}} (j_{34} j_{13} j_{14} \tau_2)_{m_{34} m_{13} m_{14}} . \end{aligned}$$

We can now use the symmetry properties of the 3j-symbols to write this as:

$$\begin{aligned} \text{L.H.S.} = & (j_{12} j_{13} j_{23} \tau_4)_{m_{12} m_{13} m_{23}} (j_{24} j_{34} j_{23} \tau_1)_{m_{24} m_{34} m_{23}} \\ & \times (j_{12} j_{24} j_{14} \tau_3)_{m_{12} m_{24} m_{14}} (j_{13} j_{34} j_{14} \tau_2)_{m_{13} m_{34} m_{14}} \\ & \times f_c [j_{24} j_{34} j_{23} \tau_1] f_c [j_{13} j_{34} j_{14} \tau_2] . \end{aligned}$$

We can now expand this expression by grouping the 3j-symbols in two pairs and using equ.(4-8). The left-hand side becomes:

$$\begin{aligned} \text{L.H.S.} = & \sum_{\substack{j, \tau, \tau', \\ \hat{j}, \hat{\tau}, \hat{\tau}'}} n_j n_{\hat{j}} (j_{24} j_{13} j \tau') (j_{12} j_{34} j \tau) (j_{12} j_{34} \hat{j} \hat{\tau}) (j_{13} j_{24} \hat{j} \hat{\tau}') \\ & \times f_c [j_{24} j_{34} j_{23} \tau_1] f_c [j_{13} j_{34} j_{14} \tau_2] \left\{ \begin{matrix} j_{12} j_{13} j_{23} & \tau_4 \tau_1 \\ j_{24} j_{34} j & \tau \tau' \end{matrix} \right\} \left\{ \begin{matrix} j_{12} j_{24} j_{14} & \tau_3 \tau_2 \\ j_{13} j_{34} \hat{j} & \hat{\tau} \hat{\tau}' \end{matrix} \right\} . \end{aligned}$$

The orthogonality of the 3j-symbols can now be used and we obtain (after using the symmetry properties of the 3j-symbols):



$$\text{L.H.S.} = \sum_{\substack{j, \tau, \tau' \\ \hat{j}, \hat{\tau}, \hat{\tau}'}} \delta_{j\hat{j}} \delta_{\tau\hat{\tau}} \delta_{\tau'\hat{\tau}'} \delta_{m\hat{m}} \delta_{m'\hat{m}'} f_c[j_{24} j_{34} j_{23} \tau_1] f_c[j_{13} j_{34} j_{14} \tau_2] \\ \times f_c[j_{24} j_{13} j \tau] \left\{ \begin{matrix} j_{12} j_{13} j_{23} \\ j_{24} j_{34} j \end{matrix} \middle| \begin{matrix} \tau_4 \tau_1 \\ \tau \tau' \end{matrix} \right\} \left\{ \begin{matrix} j_{12} j_{24} j_{14} \\ j_{13} j_{34} \hat{j} \end{matrix} \middle| \begin{matrix} \tau_3 \tau_2 \\ \hat{\tau} \hat{\tau}' \end{matrix} \right\}.$$

Performing the sums over the  $m$ 's gives a factor  $n_j$ , so that we finally obtain the required result:

$$\text{L.H.S.} = \sum_{j, \tau, \tau'} C n_j \left\{ \begin{matrix} j_{12} j_{13} j_{23} \\ j_{24} j_{34} j \end{matrix} \middle| \begin{matrix} \tau_4 \tau_1 \\ \tau \tau' \end{matrix} \right\} \left\{ \begin{matrix} j_{12} j_{24} j_{14} \\ j_{13} j_{34} j \end{matrix} \middle| \begin{matrix} \tau_3 \tau_2 \\ \tau \tau' \end{matrix} \right\},$$

where  $C = f_c[j_{24} j_{34} j_{23} \tau_1] f_c[j_{13} j_{34} j_{14} \tau_2] f_c[j_{14} j_{23} j \tau]$ .

Q.E.D.

We give below a diagram of the kind discussed at the beginning of this chapter. This diagram illustrates theorem 4.4.

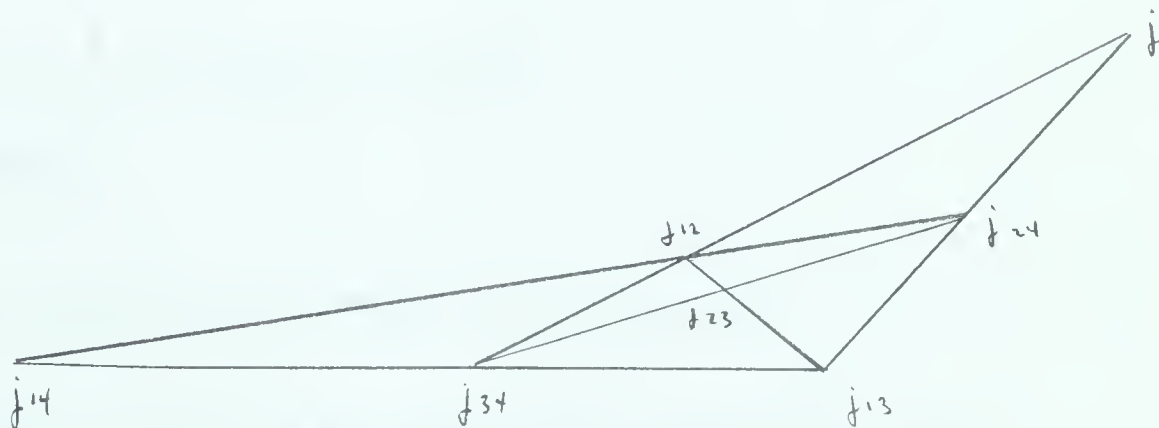


FIGURE 4.3.

Biedenharn Identity. Another interesting formula for simply reducible groups is that of Biedenharn (often called the Biedenharn identity). The generalization of the Biedenharn identity (for groups not multiplicity free) is the object of the following theorem.



Theorem 4.5:

$$\sum_{\tau_{123}} \left\{ \begin{array}{c|c} j_1 j_2 j_3 & \tau_{123} \tau_{12\bar{3}} \\ \hline j_{23} j_{13} j_{12} & \tau_{123} \tau_{1\bar{2}3} \end{array} \right\} \left\{ \begin{array}{c|c} j_1 j_2 j_3 & \tau_{123} \tau'_{12\bar{3}} \\ \hline j_{23} j_{13} j_{12} & \tau'_{123} \tau'_{1\bar{2}3} \end{array} \right\} \\ = \sum_{j, \tau, \tau', \tau''} C n_j \left\{ \begin{array}{c|c} j_{13} j'_{13} j & \tau \tau' \\ \hline j_{12} j_{12} j_1 & \tau_{123} \tau'_{123} \end{array} \right\} \left\{ \begin{array}{c|c} j_{12} j_{12} j & \tau' \tau'' \\ \hline j_{23} j_{13} j_2 & \tau'_{123} \tau_{1\bar{2}3} \end{array} \right\} \left\{ \begin{array}{c|c} j_{23} j_{23} j & \tau'' \tau \\ \hline j_{13} j_{13} j_3 & \tau_{1\bar{2}3} \tau_{12\bar{3}} \end{array} \right\}, \quad (4-14)$$

where:

$$C = f_B [j_{12} j_2 j_{23} \tau_{1\bar{2}3}] f_B [j_2 j_{12} j_{23} \tau_{1\bar{2}3}] f_c [j_2 j_{12} j_{23} \tau_{1\bar{2}3}] \\ \times f_A [j_3 j_{23} j_{13} \tau_{12\bar{3}}] f_B [j_3 j_{23} j_{13} \tau_{12\bar{3}}] f_c [j_3 j_{23} j_{13} \tau_{12\bar{3}}] \\ \times f_A [j_1 j_{13} j_{12} \tau_{123}] f_c [j_1 j_{13} j_{12} \tau_{123}] \\ \times f_A [j_{12} j_2 j_{13} \tau'_{1\bar{2}3}] f_B [j_{12} j_2 j_{13} \tau'_{1\bar{2}3}] \\ \times f_c [j_{23} j_{23} j \tau'] \\ \times f_B [j_1 j_{13} j_{12} \tau'_{123}].$$

Proof: Consider:

$$\alpha = (j_1 j_{13} j_{12} \tau_{123}) (j_1 j_{13} j_{12} \tau'_{123}) (j_2 j_{12} j_{23} \tau_{1\bar{2}3}) \\ \times (j_2 j_{12} j_{23} \tau'_{1\bar{2}3}) (j_3 j_{23} j_{13} \tau_{12\bar{3}}) (j_3 j_{23} j_{13} \tau_{1\bar{2}3}).$$

Using the symmetry properties of the 3j-symbols we obtain:

$$\alpha = C_1 (j_1 j_{13} j_{12} \tau_{123}) (j_{23} j_2 j_{12} \tau_{1\bar{2}3}) \\ \times (j_1 j_{13} j_{12} \tau_{1\bar{2}3}) (j_{23} j_2 j_{12} \tau'_{1\bar{2}3}) \\ \times (j_3 j_{23} j_{13} \tau_{12\bar{3}}) (j_3 j_{23} j_{13} \tau_{1\bar{2}3}),$$



$$\text{where } C_1 = f_A [j_2 j_{12} j_{23} \tau_{123}] f_c [j_2 j_{23} j_{12} \tau_{123}] \\ \times f_A [j_2 j_{12} j_{23} \tau'_{123}] f_c [j_2 j_{23} j_{12} \tau'_{123}] .$$

The first two lines above can be expanded and:

$$\alpha = \sum_{\substack{j, \tau, \tau', \\ j', \tau', \tau''}} C_1 n_j n_{j'} (j_1 j_2 j \tau) (j_{23} j_{13} j \tau') (j_1 j_2 j' \tau') (j_{23} j_{13} j' \tau'') \\ \times \left\{ \begin{matrix} j_1 j_2 j & \tau \tau' \\ j_{23} j_{13} j_{12} & \tau_{123} \tau'_{123} \end{matrix} \right\} \left\{ \begin{matrix} j_1 j_2 j & \tau \tau'' \\ j_{23} j_{13} j_{12} & \tau'_{123} \tau''_{123} \end{matrix} \right\} \\ \times (j_3 j_{23} j_{13} \tau_{123}) (j_3 j_{23} j_{13} \tau'_{123}) .$$

Using the orthogonality properties of the 3j-symbols we obtain:

$$\alpha = \sum_{j, \tau, \tau', \tau''} C_1 n_j (j_{23} j_{13} j \tau') (j_{23} j_{13} j \tau'') (j_3 j_{23} j_{13} \tau_{123}) (j_3 j_{23} j_{13} \tau'_{123}) \\ \times \left\{ \begin{matrix} j_1 j_2 j & \tau \tau' \\ j_{23} j_{13} j_{12} & \tau_{123} \tau'_{123} \end{matrix} \right\} \left\{ \begin{matrix} j_1 j_2 j & \tau \tau'' \\ j_{23} j_{13} j_{12} & \tau'_{123} \tau''_{123} \end{matrix} \right\}$$

and

$$\alpha = \sum_{j, \tau, \tau', \tau''} C_1 C_2 n_j (j_{23} j_{13} j \tau') (j_{23} j_{13} j \tau_{123}) (j_{23} j_{13} j \tau'') (j_{23} j_{13} j \tau'_{123}) \\ \times \left\{ \begin{matrix} j_1 j_2 j & \tau \tau' \\ j_{23} j_{13} j_{12} & \tau_{123} \tau'_{123} \end{matrix} \right\} \left\{ \begin{matrix} j_1 j_2 j & \tau \tau'' \\ j_{23} j_{13} j_{12} & \tau'_{123} \tau''_{123} \end{matrix} \right\} ,$$

where :

$$C_2 = f_A [j_3 j_{23} j_{13} \tau_{123}] f_B [j_3 j_{13} j_{23} \tau_{123}] f_A [j_3 j_{23} j_{13} \tau'_{123}] f_B [j_3 j_{13} j_{23} \tau'_{123}] .$$

The orthogonality of the 3j-symbols now gives:

$$\alpha = \sum_{j, \tau, \tau', \tau''} C_1 C_2 \left( \frac{1}{n_j} \right) \delta_{j j_3} \delta_{\tau \tau_{123}} \delta_{\tau' \tau'_{123}} \delta_{\tau'' \tau''_{123}} \delta_{m m_3} \delta_{m m_3} \\ \times \left\{ \begin{matrix} j_1 j_2 j & \tau \tau' \\ j_{23} j_{13} j_{12} & \tau_{123} \tau'_{123} \end{matrix} \right\} \left\{ \begin{matrix} j_1 j_2 j & \tau \tau'' \\ j_{23} j_{13} j_{12} & \tau'_{123} \tau''_{123} \end{matrix} \right\} .$$



The sums over  $m$  and  $m_3$  give a factor  $n_j$  and we have:

$$\alpha = \sum_{\tau_{123}} C_1 C_2 \left\{ \begin{matrix} j_1 j_2 j_3 \\ j_{23} j_{13} j_{12} \end{matrix} \middle| \begin{matrix} \tau_{123} \tau_{12\bar{3}} \\ \tau_{1\bar{2}3} \tau_{1\bar{2}\bar{3}} \end{matrix} \right\} \left\{ \begin{matrix} j_1 j_2 j_3 \\ j_{23} j_{13} j_{12} \end{matrix} \middle| \begin{matrix} \tau_{123} \tau_{12\bar{3}} \\ \tau'_{1\bar{2}3} \tau'_{1\bar{2}\bar{3}} \end{matrix} \right\},$$

where we have set  $\tau = \tau_{123}$  which is just a dummy variable. Thus, we have obtained the left-hand side of the Biedenharn identity. On the other hand, we can write:

$$\alpha = C_3 \left( j_{13} j_{12} j_1 \tau_{123} \right) \left( j'_{12} j'_{13} j_1 \tau'_{123} \right) \left( j_{23} j_{12} j_2 \tau_{1\bar{2}3} \right) \\ \times \left( j_{12} j'_{13} j_2 \tau'_{1\bar{2}3} \right) \left( j_{13} j_{23} j_3 \tau_{12\bar{3}} \right) \left( j_{23} j'_{13} j_3 \tau'_{12\bar{3}} \right),$$

where  $C_3 = f_c [j_1 j_{13} j_{12} \tau_{123}] f_A [j_{13} j_1 j_{12} \tau_{123}] f_B [j_1 j_{13} j_{12} \tau_{1\bar{2}3}]$

$$\times f_B [j_2 j_{12} j_{23} \tau_{1\bar{2}3}] f_c [j_2 j_{12} j_{23} \tau'_{1\bar{2}3}] f_A [j_{12} j_2 j_{23} \tau'_{1\bar{2}3}] \\ \times f_B [j_3 j_{23} j_{13} \tau_{12\bar{3}}] f_c [j_3 j_{23} j_{13} \tau'_{12\bar{3}}] f_A [j_{23} j_3 j_{13} \tau'_{12\bar{3}}].$$

We can now expand this expression:

$$\alpha = \sum_{\substack{j, \tau, \tau', \\ \hat{j}, \hat{\tau}, \hat{\tau}', \\ \tilde{j}, \tilde{\tau}, \tilde{\tau}'}} C_3 n_j n_{\tilde{j}} n_{\tilde{j}} (j_{13} j_{13}' j \tau) (j'_{12} j_{12} j \tau') (j_{12} j_{12} \hat{j} \hat{\tau}) (j_{23} j_{23} \hat{j} \hat{\tau}') \\ \times (j_{23} j_{23} \tilde{j} \tilde{\tau}) (j_{13} j_{13}' \tilde{j} \tilde{\tau}') \left\{ \begin{matrix} j_{13} j_{13}' j \\ j_{12} j_{12} j_1 \end{matrix} \middle| \begin{matrix} \tau \tau' \\ \tau_{123} \tau_{1\bar{2}3} \end{matrix} \right\} \left\{ \begin{matrix} j'_{12} j_{12} \hat{j} \\ j_{23} j_{23} j_2 \end{matrix} \middle| \begin{matrix} \hat{\tau} \hat{\tau}' \\ \tau'_{1\bar{2}3} \tau_{1\bar{2}\bar{3}} \end{matrix} \right\} \left\{ \begin{matrix} j_{23} j_{23} \tilde{j} \\ j_{13} j_{13}' j_3 \end{matrix} \middle| \begin{matrix} \tilde{\tau} \tilde{\tau}' \\ \tau_{1\bar{2}\bar{3}} \tau_{12\bar{3}} \end{matrix} \right\}.$$

We can now use the orthogonality of the  $3j$ -symbols.

$$\alpha = \sum_{\substack{j, \tau, \tau', \\ \hat{j}, \hat{\tau}, \hat{\tau}', \\ \tilde{j}, \tilde{\tau}, \tilde{\tau}'}} C_3 C_4 \delta_{j\tilde{j}} \delta_{j\hat{j}} \delta_{\tilde{j}\hat{j}} \delta_{m\tilde{m}} \delta_{m\hat{m}} \delta_{\tilde{m}\hat{m}} \delta_{\tau\tilde{\tau}} \delta_{\hat{\tau}\tau'} \delta_{\tilde{\tau}\hat{\tau}'} \\ \times \left\{ \begin{matrix} j_{13} j_{13}' j \\ j_{12} j_{12} j_1 \end{matrix} \middle| \begin{matrix} \tau \tau' \\ \tau_{123} \tau_{1\bar{2}3} \end{matrix} \right\} \left\{ \begin{matrix} j_{12} j_{12} \hat{j} \\ j_{23} j_{23} j_2 \end{matrix} \middle| \begin{matrix} \hat{\tau} \hat{\tau}' \\ \tau'_{1\bar{2}3} \tau_{1\bar{2}\bar{3}} \end{matrix} \right\} \left\{ \begin{matrix} j_{23} j_{23} \tilde{j} \\ j_{13} j_{13}' j_3 \end{matrix} \middle| \begin{matrix} \tilde{\tau} \tilde{\tau}' \\ \tau_{1\bar{2}\bar{3}} \tau_{12\bar{3}} \end{matrix} \right\},$$



where  $C_4 = \{c[j_{23} j'_{23} \hat{j} \hat{\tau}']\}$ . This can be reduced to:

$$\alpha = \sum_{j, \tau, \tau', \tau''} C_3 C_4 n_j \left\{ \begin{matrix} j_{13} j'_{13} j \\ j_{12} j'_{12} j_1 \end{matrix} \middle| \begin{matrix} \tau & \tau' \\ \tau_{123} & \tau'_{123} \end{matrix} \right\} \left\{ \begin{matrix} j_{12} j'_{12} j \\ j_{23} j'_{23} j_2 \end{matrix} \middle| \begin{matrix} \tau' & \tau'' \\ \tau'_{123} & \tau''_{123} \end{matrix} \right\} \left\{ \begin{matrix} j_{23} j'_{23} j \\ j_{13} j'_{13} j_3 \end{matrix} \middle| \begin{matrix} \tau'' & \tau \\ \tau''_{123} & \tau_{123} \end{matrix} \right\},$$

where we have set  $\hat{\tau}' = \tau''$ . This is the right-hand side of equ.(4-14) and the proof is complete. With table 2-1 and the symmetries of the  $f$  functions, it is a simple matter to show that  $C = C_1 C_2 C_3 C_4$ . The diagram for Biedenharn identity is shown below.

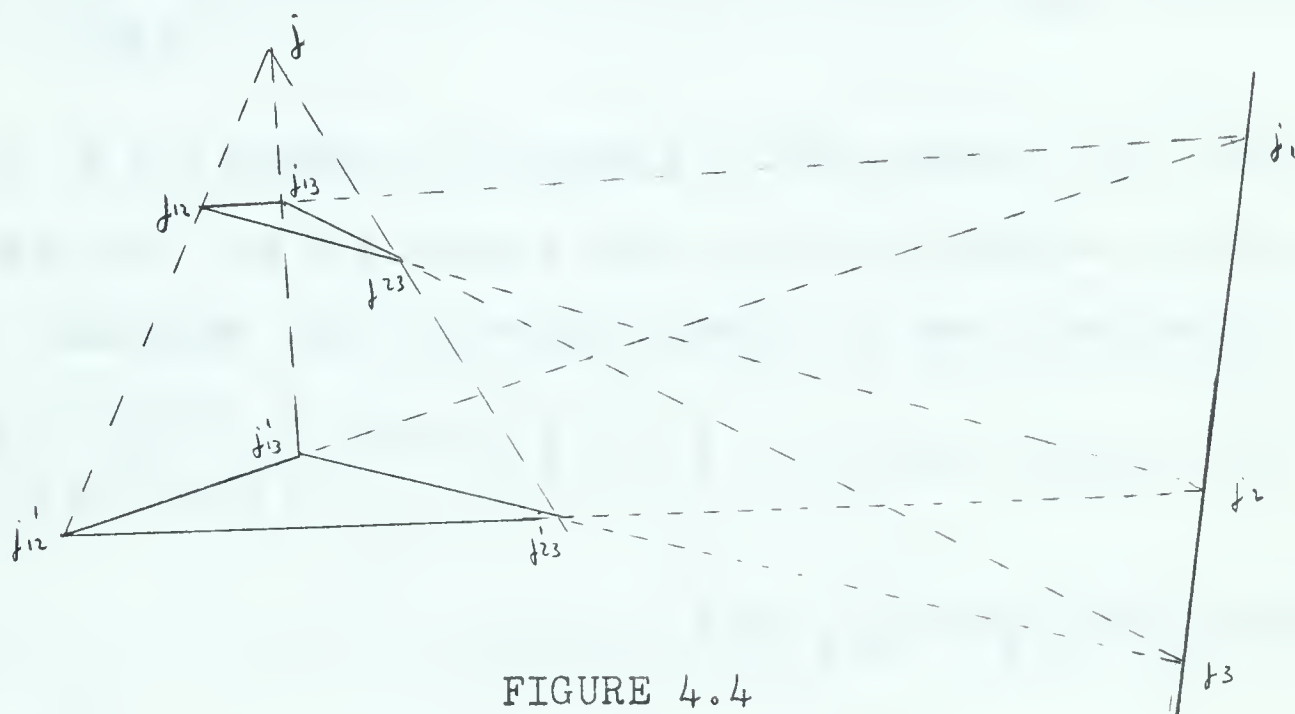


FIGURE 4.4

We see that figure 4.4 expresses the validity of a theorem of plane geometry due to Desargues: "if two triangles are so situated that the three points of intersection of corresponding sides of the two triangles are collinear, then the joins of corresponding vertices of the two triangles are concurrent."



Group Sums. We shall now prove a few theorems on group sums. We shall see that certain functions of 6j-symbols can be expressed in terms of characters. These formulae were obtained by Wigner<sup>2</sup> for simply reducible groups.

Theorem 4.6:

$$\sum_{\tau_1, \tau_2, \tau_3, \tau_4} \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\}^2$$

$$= (1/g^3) \sum_{R_1, R_2, R_3} \chi^{j_{12}}(R_1) \chi^{j_{13}}(R_2) \chi^{j_{23}}(R_3) \chi^{j_{34}}(R_2 R_3^{-1}) \chi^{j_{24}}(R_3 R_1^{-1}) \chi^{j_{14}}(R_1 R_2^{-1}), \quad (4-15)$$

where g is the number of elements in the group, i.e. its order.

The sums over the R's extend over all the elements of the group.

Proof: Consider the following product of two 6j-symbols:

$$\left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\} \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4' & \tau_1' \\ j_{34} & j_{24} & j_{14} & \tau_3' & \tau_2' \end{array} \right\} = (j_{12} j_{13} j_{23} \tau_4) (j_{12} j_{24} j_{14} \tau_3) (j_{34} j_{24} j_{23} \tau_1)$$

$$\times (j_{34} j_{13} j_{14} \tau_2) (j_{12} j_{13} j_{23} \tau_4') (j_{12} j_{24} j_{14} \tau_3') (j_{34} j_{24} j_{23} \tau_1') (j_{34} j_{13} j_{14} \tau_2')$$

Therefore, we can write:

$$\sum_{\tau_1, \tau_2, \tau_3, \tau_4} \left\{ \begin{array}{ccc|cc} j_{12} & j_{13} & j_{23} & \tau_4 & \tau_1 \\ j_{34} & j_{24} & j_{14} & \tau_3 & \tau_2 \end{array} \right\}^2$$

$$= \sum_{\substack{\tau_1, \tau_2, \\ \tau_3, \tau_4}} (j_{12} j_{13} j_{23} \tau_4) m_{12} m_{13} m_{23} (j_{12} j_{24} j_{14} \tau_3) m_{12} m_{24} m_{14} (j_{34} j_{24} j_{23} \tau_1) m_{34} m_{24} m_{23} (j_{34} j_{13} j_{14} \tau_2) m_{34} m_{24} m_{12}$$

$$\times (j_{12} j_{13} j_{23} \tau_4') m_{12}' m_{13}' m_{23}' (j_{12} j_{24} j_{14} \tau_3') m_{12}' m_{24}' m_{14}' (j_{34} j_{24} j_{23} \tau_1') m_{34}' m_{24}' m_{23}' (j_{34} j_{13} j_{14} \tau_2') m_{34}' m_{24}' m_{12}'$$

Each pair of 3j-symbols can be expressed in terms of the D's with equ.(2-14) or equ.(A 6)' and the expression above becomes:



$$\begin{aligned}
 & (1/q^4) \sum_{R,S,T,U} D_{m_{13}m'_{13}}^{j_{23}}(R) D_{m_{12}m'_{12}}^{j_{12}}(R) D_{m_{13}m'_{13}}^{j_{13}}(R) \\
 & \times D_{m_{12}m'_{12}}^{j_{12}}(S) D_{m_{24}m'_{24}}^{j_{24}}(S) D_{m_{14}m'_{14}}^{j_{14}}(S) \\
 & \times D_{m_{34}m'_{34}}^{j_{34}}(T) D_{m_{13}m'_{13}}^{j_{13}}(T) D_{m_{14}m'_{14}}^{j_{14}}(T) \\
 & \times D_{m_{34}m'_{34}}^{j_{34}}(U) D_{m_{24}m'_{24}}^{j_{24}}(U) D_{m_{23}m'_{23}}^{j_{23}}(U).
 \end{aligned}$$

Performing some of the multiplications, this becomes:

$$\begin{aligned}
 & (1/q^4) \sum_{R,S,T,U} D_{m_{12}m'_{12}}^{j_{12}}(RS) D_{m_{13}m'_{13}}^{j_{13}}(RT) D_{m_{14}m'_{14}}^{j_{14}}(SU) \\
 & \times D_{m_{34}m'_{34}}^{j_{34}}(TU) D_{m_{23}m'_{23}}^{j_{23}}(RU^{-1}) D_{m_{14}m'_{14}}^{j_{14}}(ST^{-1}),
 \end{aligned}$$

where, for the last two terms, we used the fact that the matrices are orthogonal. By definition of the character of a representation the above expression is simply:

$$(1/q^4) \sum_{R,S,T,U} \chi^{j_{12}}(RS) \chi^{j_{13}}(RT) \chi^{j_{14}}(SU) \chi^{j_{34}}(TU) \chi^{j_{23}}(RU^{-1}) \chi^{j_{14}}(ST^{-1}).$$

This can be simplified somewhat. Setting  $R_1 = RS$ ,  $R_2 = RT$ , and  $R_3 = RU^{-1}$ , we obtain for the above expression:

$$(1/q^4) \sum_{\substack{R_1, R_2, R_3 \\ R}} \chi^{j_{12}}(R_1) \chi^{j_{13}}(R_2) \chi^{j_{23}}(R_3) \chi^{j_{14}}(R_3 R_1^{-1}) \chi^{j_{34}}(R_2 R_3^{-1}) \chi^{j_{14}}(R_1 R_2^{-1}),$$

where we have used the fact that  $\text{tr}AB = \text{tr}BA$  and that the matrices are orthogonal (this last condition is sufficient but not necessary). The expression inside the summations is independent of  $R$ . The



sum over R then simply gives a factor g which cancels one of the g's in the denominator. This completes the proof of theorem 4.6. A second group sum is given in the next theorem.

Theorem 4.7:

$$\sum_{\tau, \tau'} \left\{ \begin{matrix} j_1 & j_2 & j' \\ j_1 & j_2 & j \end{matrix} \middle| \begin{matrix} \tau' & \tau' \\ \tau & \tau \end{matrix} \right\} = (1/g^2) \sum_{R, S} \chi^{j_1}(R) \chi^{j_2}(S) \chi^{j_1}(S^{-1}R) \chi^{j_2}(SR). \quad (4-16)$$

Proof: Consider the left-hand side of the above expression. By definition, it can be written as:

$$\text{L.H.S.} = \sum_{\tau, \tau'} (j_1 j_2 j \tau)_{m_1 m_2 m} (j_1 j_2 j \tau)_{m'_1 m'_2 m} (j_1 j_2 j' \tau')_{m_1 m'_2 m'} (j_1 j_2 j' \tau')_{m'_1 m_2 m'}.$$

Grouping the 3j-symbols in pairs, we can express each pair in terms of representation matrices as in the previous theorem. We obtain:

$$\text{L.H.S.} = (1/g^2) \sum_{R, S} D_{mm}^{j_1}(S) D_{m_1 m'_1}^{j_2}(S) D_{m_1 m'_1}^{j_2}(S) D_{m'_1 m'}^{j_1}(R) D_{m_1 m'_1}^{j_1}(R) D_{m'_1 m_2}^{j_2}(R)$$

and

$$\text{L.H.S.} = (1/g^2) \sum_{R, S} D_{mm}^{j_1}(S) D_{m'_1 m'}^{j_1}(R) D_{m'_1 m'_1}^{j_1}(S^{-1}R) D_{m'_1 m_2}^{j_2}(RS).$$

By definition of the character of an irreducible representation we have:



$$L.H.S. = (1/g^2) \sum_{R, S} \chi^{\dagger}(S) \chi^{\dagger}(R) \chi^{\dagger}(S^{-1}R) \chi^{\dagger}(RS).$$

Q.E.D.

Theorem 4.8:

$$\begin{aligned} & \sum_{\substack{\tau_{12}, \tau'_{12}, \tau''_{12}, \\ \tau_{23}, \tau'_{23}, \tau''_{23}}} \left\{ \begin{matrix} j' & j_1 & j_2 \\ j & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau'_{12} & \tau_{23} \\ \tau'_{23} & \tau_{12} \end{matrix} \right\} \left\{ \begin{matrix} j' & j_1 & j_2 \\ j'' & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau'_{12} & \tau''_{23} \\ \tau'_{23} & \tau''_{12} \end{matrix} \right\} \left\{ \begin{matrix} j & j_1 & j_2 \\ j'' & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau_{12} & \tau''_{23} \\ \tau_{23} & \tau''_{12} \end{matrix} \right\} \\ &= (1/g^4) \sum_{P, R, R', R''} \chi^{\dagger}(PR') \chi^{\dagger}(PR'R'P) \chi^{\dagger}(RPR'') \chi^{\dagger}(R) \chi^{\dagger}(R') \chi^{\dagger}(R''). \end{aligned} \quad (4-17)$$

Proof: Consider the following expression:

$$\begin{aligned} & \left\{ \begin{matrix} j' & j_1 & j_2 \\ j & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau'_{12} & \tau_{23} \\ \tau'_{23} & \tau_{12} \end{matrix} \right\} \left\{ \begin{matrix} \hat{j}' & j_1' & \hat{j}'_2 \\ j'' & j_3' & j_2' \end{matrix} \middle| \begin{matrix} \tau'_{12} & \tau''_{23} \\ \tau'_{23} & \tau''_{12} \end{matrix} \right\} \left\{ \begin{matrix} \hat{j} & j_1'' & \hat{j}''_2 \\ \hat{j}'' & j_3'' & \hat{j}''_2 \end{matrix} \middle| \begin{matrix} \tau_{12} & \tau''_{23} \\ \tau_{23} & \tau''_{12} \end{matrix} \right\} \\ &= (j' j_1 j_2 \tau'_{12})_{m' m_1 m_2} (\hat{j}' j_1' \hat{j}'_2 \tau'_{12})_{\hat{m}' m_1' \hat{m}'_2} (j j_3 \hat{j}_2 \tau_{23})_{m m_3 \hat{m}_2} (\hat{j} j_3' j_2' \tau_{23})_{\hat{m} m_3' \hat{m}_2'} \\ & \times (j j_1 \hat{j}_2 \tau_{12})_{m m_1 \hat{m}_2} (\hat{j} j_1'' \hat{j}''_2 \tau_{12})_{\hat{m} m_1'' \hat{m}_2''} (j'' j_3' j_2' \tau'_{12})_{m'' m_3' m_2'} (\hat{j}'' j_1'' \hat{j}''_2 \tau'_{12})_{\hat{m}'' m_1'' \hat{m}_2''} \\ & \times (j j_3 j_2 \tau_{23})_{m m_3 m_2} (\hat{j} j_3'' \hat{j}''_2 \tau_{23})_{\hat{m} m_3'' \hat{m}_2''} (j'' j_3' \hat{j}'_2 \tau'_{23})_{m'' m_3' \hat{m}_2'} (\hat{j}'' j_3'' \hat{j}''_2 \tau'_{23})_{\hat{m}'' m_3'' \hat{m}_2''}. \end{aligned}$$

It follows that:

$$\begin{aligned} & \sum_{\substack{\tau_{12}, \tau'_{12}, \tau''_{12}, \\ \tau_{23}, \tau'_{23}, \tau''_{23}}} \left\{ \begin{matrix} j' & j_1 & j_2 \\ j & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau'_{12} & \tau_{23} \\ \tau'_{23} & \tau_{12} \end{matrix} \right\} \left\{ \begin{matrix} j' & j_1 & j_2 \\ j'' & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau'_{12} & \tau''_{23} \\ \tau'_{23} & \tau''_{12} \end{matrix} \right\} \left\{ \begin{matrix} j & j_1 & j_2 \\ j'' & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau_{12} & \tau''_{23} \\ \tau_{23} & \tau''_{12} \end{matrix} \right\} \\ &= (1/g^4) \sum_{\substack{Q, S, T, \\ U, V, W}} D_{m_1' \hat{m}_1'}^{\dagger}(Q) D_{m_1 m_1'}^{\dagger}(Q) D_{m_2 \hat{m}_2'}^{\dagger}(Q) D_{m_1' \hat{m}_1'}^{\dagger}(S) D_{m_3 m_3'}^{\dagger}(S) D_{\hat{m}_2 m_2'}^{\dagger}(S) \\ & \times D_{\hat{m}_1 m_1}^{\dagger}(T) D_{m_1'' m_1'}^{\dagger}(T) D_{\hat{m}_2' m_2}^{\dagger}(T) D_{m_1'' \hat{m}_1''}^{\dagger}(U) D_{m_1' m_1''}^{\dagger}(U) D_{\hat{m}_2'' m_2''}^{\dagger}(U) \\ & \times D_{\hat{m}_1 m_1}^{\dagger}(V) D_{m_3'' m_3}^{\dagger}(V) D_{\hat{m}_2'' m_2}^{\dagger}(V) D_{m_1'' \hat{m}_1''}^{\dagger}(W) D_{m_3' m_3''}^{\dagger}(W) D_{\hat{m}_2' m_2'}^{\dagger}(W), \end{aligned}$$



and using the orthogonality of the matrices this becomes:

$$\begin{aligned}
 & (1/g^6) \sum_{\substack{Q,S,T, \\ U,V,W}} D_{m_1 m_1}^{j_1'}(QS^{-1}) D_{m_1 m_1}^{j_1'}(QUT) D_{m_2 m_2}^{j_2'}(QWT SUV) \\
 & \quad \times D_{m_3 m_3}^{j_3'}(SWV) D_{\hat{m} \hat{m}}^{j_4'}(TV^{-1}) D_{m'' m''}^{j_5'}(UW^{-1}) \\
 & = (1/g^6) \sum_{\substack{Q,S,T, \\ U,V,W}} \chi^{j_1'}(QS^{-1}) \chi^{j_1'}(QUT) \chi^{j_2'}(QWT SUV) \\
 & \quad \times \chi^{j_3'}(SWV) \chi^{j_4'}(TV^{-1}) \chi^{j_5'}(UW^{-1}).
 \end{aligned}$$

We now introduce the new variables:

$$R = V^{-1}T, R' = SQ^{-1}, R'' = SUW^{-1}S^{-1}, P = T^{-1}U^{-1}S^{-1},$$

and obtain:

$$\begin{aligned}
 (QUT)^{-1} &= T^{-1}U^{-1}S^{-1}. \quad SQ^{-1} = PR' \\
 (SWV)^{-1} &= V^{-1}T.T^{-1}U^{-1}S^{-1}. \quad SUW^{-1}S^{-1} = RPR'' \\
 (QWTSUV)^{-1} &= V^{-1}T.(T^{-1}U^{-1}S^{-1})^2. \quad SUW^{-1}S^{-1}. \quad SQ^{-1} = RP^2R''R'.
 \end{aligned}$$

Hence, we have:

$$\begin{aligned}
 & \sum_{\substack{\tau_{12}, \tau_{12}', \tau_{12}'', \\ \tau_{13}, \tau_{13}', \tau_{13}''}} \left\{ \begin{matrix} j_1' & j_1 & j_2 \\ & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau_{12}' & \tau_{13} \\ \tau_{13}' & \tau_{12} \end{matrix} \right\} \left\{ \begin{matrix} j_1' & j_1 & j_2 \\ & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau_{12}' & \tau_{13}'' \\ \tau_{13}' & \tau_{12}'' \end{matrix} \right\} \left\{ \begin{matrix} j_1 & j_1 & j_2 \\ & j_3 & j_2 \end{matrix} \middle| \begin{matrix} \tau_{12} & \tau_{13}'' \\ \tau_{13} & \tau_{12}'' \end{matrix} \right\} \\
 & = (1/g^4) \sum_{P, R, R', R''} \chi^{j_1'}(R) \chi^{j_1'}(R') \chi^{j_2'}(R'') \chi^{j_3'}(PR') \chi^{j_4'}(RP^2R''R') \chi^{j_5'}(RPR'') \\
 & = (1/g^4) \sum_{P, R, R', R''} \chi^{j_1'}(PR') \chi^{j_2'}(PR''R'RP) \chi^{j_3'}(RPR'') \chi^{j_4'}(R) \chi^{j_5'}(R') \chi^{j_6'}(R'').
 \end{aligned}$$

Q.E.D.



Theorem 4.9:

$$\begin{aligned}
 & \sum_{\substack{\tau_1, \tau_2, \tau_3, \tau_4 \\ \tau'_1, \tau'_2, \tau'_3, \tau'_4}} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{12} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau'_1 \\ \tau_3 & \tau'_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{13} & j_{14} \end{matrix} \middle| \begin{matrix} \tau'_4 & \tau'_3 \\ \tau'_2 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{23} \end{matrix} \middle| \begin{matrix} \tau'_4 & \tau_1 \\ \tau'_1 & \tau'_3 \end{matrix} \right\} \\
 &= (1/g^5) \sum_{\substack{R, S, \\ R_1, R_2, R_3}} \chi^{j_{12}}(R) \chi^{j_{12}}(S^{-1} R_1 R_2) \chi^{j_{23}}(S R R_2 R_3) \chi^{j_{13}}(R^{-1} S R_3 R_1) \\
 & \quad \times \chi^{j_{34}}(R_3) \chi^{j_{24}}(R_2) \chi^{j_{14}}(R_1). \tag{4-18}
 \end{aligned}$$

Proof: By definition of the 6j-symbols, we have:

$$\begin{aligned}
 & \left\{ \begin{matrix} j_{12}^a & j_{13}^a & j_{23}^a \\ j_{34}^a & j_{24}^a & j_{14}^a \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12}^b & j_{13}^b & j_{23}^b \\ j_{12}^b & j_{24}^b & j_{14}^b \end{matrix} \middle| \begin{matrix} \tau_4 & \tau'_1 \\ \tau_3 & \tau'_2 \end{matrix} \right\} \\
 & \quad \times \left\{ \begin{matrix} j_{12}^c & j_{13}^c & j_{23}^c \\ j_{34}^c & j_{13}^c & j_{14}^c \end{matrix} \middle| \begin{matrix} \tau'_4 & \tau'_3 \\ \tau'_2 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12}^d & j_{13}^d & j_{23}^d \\ j_{34}^d & j_{24}^d & j_{23}^d \end{matrix} \middle| \begin{matrix} \tau'_4 & \tau_1 \\ \tau'_1 & \tau'_3 \end{matrix} \right\} \\
 &= (j_{12}^a j_{13}^a j_{23}^a \tau_4) m_{12}^a m_{13}^a m_{23}^a (j_{12}^b j_{13}^b j_{23}^b \tau_4) m_{12}^b m_{13}^b m_{23}^b \\
 & \quad \times (j_{12}^a j_{24}^a j_{14}^a \tau_3) m_{12}^a m_{24}^a m_{14}^a (j_{12}^b j_{14}^b j_{14}^b \tau_3) m_{12}^b m_{24}^b m_{14}^b \\
 & \quad \times (j_{12}^c j_{13}^c j_{23}^c \tau'_4) m_{12}^c m_{13}^c m_{23}^c (j_{12}^d j_{13}^d j_{23}^d \tau'_4) m_{12}^d m_{13}^d m_{23}^d \\
 & \quad \times (j_{34}^a j_{13}^a j_{14}^a \tau_2) m_{34}^a m_{13}^a m_{14}^a (j_{34}^c j_{13}^c j_{14}^c \tau_2) m_{34}^c m_{13}^c m_{14}^c \\
 & \quad \times (\hat{j}_{12}^b j_{13}^b j_{14}^b \tau'_2) \hat{m}_{12}^b m_{13}^b m_{14}^b (j_{12}^c \hat{j}_{13}^c j_{14}^c \tau'_2) m_{12}^c \hat{m}_{13}^c m_{14}^c \\
 & \quad \times (j_{34}^a j_{24}^a j_{23}^a \tau_1) m_{34}^a m_{24}^a m_{23}^a (j_{34}^d j_{24}^d j_{23}^d \tau_1) m_{34}^d m_{24}^d m_{23}^d \\
 & \quad \times (\hat{j}_{12}^b j_{24}^b j_{23}^b \tau'_1) \hat{m}_{12}^b m_{24}^b m_{23}^b (j_{12}^d j_{24}^d \hat{j}_{23}^d \tau'_1) m_{12}^d m_{24}^d \hat{m}_{23}^d \\
 & \quad \times (j_{34}^a j_{13}^a \hat{j}_{23}^a \tau'_3) m_{34}^a m_{13}^a \hat{m}_{23}^a (j_{34}^c \hat{j}_{13}^c j_{23}^c \tau'_3) m_{34}^c \hat{m}_{13}^c m_{23}^c.
 \end{aligned}$$



As before, each pair can be expanded in terms of the D matrices and we obtain:

$$\begin{aligned}
 & \sum_{\substack{\tau_1, \tau_2, \tau_3, \tau_4, \\ \tau'_1, \tau'_2, \tau'_3, \tau'_4}} \left\{ \begin{matrix} j_{12} j_{13} j_{23} \\ j_{34} j_{24} j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 \tau_1 \\ \tau_3 \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} j_{13} j_{23} \\ j_{12} j_{24} j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 \tau'_1 \\ \tau_3 \tau'_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} j_{13} j_{23} \\ j_{34} j_{13} j_{14} \end{matrix} \middle| \begin{matrix} \tau'_4 \tau'_3 \\ \tau'_2 \tau'_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} j_{13} j_{23} \\ j_{34} j_{24} j_{23} \end{matrix} \middle| \begin{matrix} \tau'_4 \tau'_1 \\ \tau'_1 \tau'_3 \end{matrix} \right\} \\
 &= (1/g^8) \sum_{\substack{P_1, P_2, P_3, P_4, \\ P_5, P_6, P_7, P_8}} D_{m_{12}^a m_{12}^b}^{j_{12}}(P_1) D_{m_{13}^a m_{13}^b}^{j_{13}}(P_1) D_{m_{23}^a m_{23}^b}^{j_{23}}(P_1) D_{m_{12}^b m_{12}^a}^{j_{12}}(P_2) D_{m_{14}^b m_{14}^a}^{j_{14}}(P_2) D_{m_{14}^b m_{14}^a}^{j_{14}}(P_2) \\
 & \quad \times D_{m_{15}^c m_{15}^d}^{j_{12}}(P_3) D_{m_{13}^c m_{13}^d}^{j_{13}}(P_3) D_{m_{23}^c m_{23}^d}^{j_{23}}(P_3) D_{m_{34}^d m_{34}^a}^{j_{34}}(P_4) D_{m_{13}^c m_{13}^a}^{j_{13}}(P_4) D_{m_{15}^c m_{15}^a}^{j_{14}}(P_4) \\
 & \quad \times D_{m_{12}^b m_{12}^c}^{j_{12}}(P_5) D_{m_{13}^b m_{13}^c}^{j_{13}}(P_5) D_{m_{14}^b m_{14}^c}^{j_{14}}(P_5) D_{m_{34}^d m_{34}^a}^{j_{34}}(P_6) D_{m_{14}^d m_{14}^a}^{j_{14}}(P_6) D_{m_{23}^d m_{23}^a}^{j_{23}}(P_6) \\
 & \quad \times D_{m_{12}^b m_{12}^d}^{j_{12}}(P_7) D_{m_{14}^b m_{14}^d}^{j_{14}}(P_7) D_{m_{23}^b m_{23}^d}^{j_{23}}(P_7) D_{m_{34}^d m_{34}^c}^{j_{34}}(P_8) D_{m_{13}^d m_{13}^c}^{j_{13}}(P_8) D_{m_{23}^d m_{23}^c}^{j_{23}}(P_8),
 \end{aligned}$$

and performing the multiplications we have:

$$\begin{aligned}
 \text{R.H.S.} &= (1/g^8) \sum_{P_1, \dots, P_8} D_{m_{12}^a m_{12}^a}^{j_{12}}(P_1, P_2) D_{m_{12}^b m_{12}^b}^{j_{12}}(P_3, P_7^{-1} P_5) D_{m_{13}^a m_{13}^a}^{j_{13}}(P_1, P_5, P_8^{-1} P_3^{-1} P_4) \\
 & \quad \times D_{m_{23}^a m_{23}^a}^{j_{23}}(P_1, P_7, P_8, P_3, P_6) D_{m_{14}^a m_{14}^a}^{j_{14}}(P_2^{-1} P_7, P_6) D_{m_{14}^b m_{14}^b}^{j_{14}}(P_2, P_4^{-1} P_5^{-1}) D_{m_{34}^c m_{34}^c}^{j_{34}}(P_4, P_6^{-1} P_8) \\
 &= (1/g^8) \sum_{P_1, \dots, P_8} \chi^{j_{14}}(P_1, P_2) \chi^{j_{12}}(P_3, P_7^{-1} P_5) \chi^{j_{13}}(P_1, P_5, P_8^{-1} P_3^{-1} P_4) \chi^{j_{23}}(P_1, P_7, P_8, P_3, P_6) \\
 & \quad \times \chi^{j_{14}}(P_2^{-1} P_7, P_6) \chi^{j_{14}}(P_2, P_4^{-1} P_5^{-1}) \chi^{j_{34}}(P_4, P_6^{-1} P_8).
 \end{aligned}$$

We now set:

$$R = P_1 P_2, \quad S = P_4^{-1} P_3 P_6, \quad R_1 = P_4^{-1} P_5^{-1} P_2, \quad R_2 = P_2^{-1} P_7 P_6, \quad R_3 = P_6^{-1} P_8 P_4.$$



It follows that:

$$P_6^{-1} (P_3^{-1} P_5^{-1} P_7) P_6 = (P_6^{-1} P_3^{-1} P_4) (P_4^{-1} P_5^{-1} P_2) (P_2^{-1} P_7 P_1) = S^{-1} R_1 R_2,$$

$$(P_4^{-1} P_3 P_6) (P_1 P_7 P_8 P_3 P_6) (P_4^{-1} P_3 P_6)^{-1} \\ = (P_4^{-1} P_3 P_6) (P_1 R_2) (P_2^{-1} P_7 P_1) (P_6^{-1} P_5 P_4) = S R R_2 R_3,$$

$$(P_1 P_2)^{-1} (P_4^{-1} P_3 P_5 P_5^{-1} P_1^{-1}) (P_1 P_2) \\ = (P_1 P_2)^{-1} (P_4^{-1} P_3 P_6) (P_6^{-1} P_8 P_4) (P_4^{-1} P_5^{-1} P_2) = R^{-1} S R_3 R_1,$$

and we obtain:

$$\chi^{j_{12}}(P_3 P_7^{-1} P_5) = \chi^{j_{12}}(P_7^{-1} P_5 P_3) = \chi^{j_{12}}(P_3^{-1} P_5^{-1} P_7) = \chi^{j_{12}}(S^{-1} R_1 R_2),$$

$$\chi^{j_{13}}(P_1 P_5 P_8^{-1} P_3^{-1} P_4) = \chi^{j_{13}}(P_4^{-1} P_3 P_8 P_5^{-1} P_1^{-1}) = \chi^{j_{13}}(R^{-1} S R_3 R_1),$$

$$\chi^{j_{23}}(P_1 P_7 P_8 P_3 P_6) = \chi^{j_{23}}(S R R_2 R_3),$$

so that:

$$\sum_{\substack{\tau_1, \tau_2, \tau_3, \tau_4, \\ \tau'_1, \tau'_2, \tau'_3, \tau'_4}} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{34} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau_1 \\ \tau_3 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{12} & j_{24} & j_{14} \end{matrix} \middle| \begin{matrix} \tau_4 & \tau'_1 \\ \tau_3 & \tau'_2 \end{matrix} \right\} \\ \times \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{13} & j_{14} \end{matrix} \middle| \begin{matrix} \tau'_4 & \tau'_3 \\ \tau'_1 & \tau_2 \end{matrix} \right\} \left\{ \begin{matrix} j_{12} & j_{13} & j_{23} \\ j_{34} & j_{24} & j_{23} \end{matrix} \middle| \begin{matrix} \tau'_4 & \tau_1 \\ \tau'_1 & \tau'_3 \end{matrix} \right\}$$

$$= (1/g^5) \sum_{\substack{R, S, \\ R_1, R_2, R_3}} \chi^{j_{12}}(R) \chi^{j_{12}}(S^{-1} R_1 R_2) \chi^{j_{13}}(R^{-1} S R_3 R_1) \chi^{j_{23}}(S R R_2 R_3) \\ \times \chi^{j_{34}}(R_3) \chi^{j_{24}}(R_2) \chi^{j_{14}}(R_1).$$

Q.E.D.



We now give a brief discussion of the general significance of the group sums. More details can be found in the thesis of W.T. Sharp. In the four group sums, i.e. theorems 4.6, 4.7, 4.8 and 4.9, the left-hand side is a sum over all the  $\tau$  variables in the products of the  $6j$ -symbols. Every  $\tau$  variable, summed over, appears twice in the product. Each triad also appears twice. On the right-hand side, we have a sum over the group of products of characters.

We know that the  $6j$ -symbols are invariant under an inner basis transformation. However, under an outer transformation they transform according to equ.(4-7). The right-hand sides of the group sums are certainly invariant under an outer (as well as an inner) basis transformation. The left-hand sides are also invariant under an outer transformation. We show this directly in the following theorem.

Theorem 4.10. The left-hand sides of the group sums are invariant under an outer transformation.

Proof: The left-hand sides of the group sums are of the form:

$$\sum_{\text{all } \tau} \left\{ \begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \middle| \begin{matrix} \tau_1 \\ \cdot \end{matrix} \right\} \left\{ \begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \middle| \begin{matrix} \tau_2 \\ \cdot \end{matrix} \right\} \cdots \left\{ \begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \middle| \begin{matrix} \tau_n \\ \cdot \end{matrix} \right\} \left\{ \begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \middle| \begin{matrix} \tau_1 \\ \cdot \end{matrix} \right\} \cdots \left\{ \begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix} \middle| \begin{matrix} \tau_n \\ \cdot \end{matrix} \right\} .$$

This expression is not strictly correct but it is to be interpreted symbolically. What is meant is that each  $\tau$  variable appears twice. In some cases,  $\tau_i$  and  $\tau_j$  may be in the same  $6j$ -symbol, or even the two  $\tau$  may be in the same  $6j$ -symbol. However, this proof



does not use these facts, and it is irrelevant for this proof where the  $\tau$ 's are, so long as they occur in pairs.

According to equ.(4-7) we have, since each  $\tau$  appears twice:

$$\begin{aligned} & U \sum_{\text{all } \tau} \{ \dots | \tau_i \} \dots \{ \dots | \tau_j \} \{ \dots | \tau_k \} \dots \{ \dots | \tau_l \} \\ &= \sum_{\text{all } \tau} \sum_{\substack{\text{all } \tau', \\ \text{all } \tau''}} (U_{\tau, \tau'} U_{\tau, \tau'} \dots U_{\tau, \tau'}) (U_{\tau, \tau''} U_{\tau, \tau''} \dots U_{\tau, \tau''}) \{ \dots | \tau'_i \} \dots \{ \dots | \tau'_j \} \{ \dots | \tau''_k \} \dots \{ \dots | \tau''_l \}. \end{aligned}$$

Using the orthogonality of the U matrices we obtain:

$$\text{R.H.S.} = \sum_{\substack{\text{all } \tau', \\ \text{all } \tau''}} \delta_{\tau, \tau'} \delta_{\tau, \tau''} \dots \delta_{\tau, \tau''} \{ \dots | \tau'_i \} \dots \{ \dots | \tau'_j \} \{ \dots | \tau''_k \} \dots \{ \dots | \tau''_l \}$$

and

$$\text{R.H.S.} = \sum_{\text{all } \tau'} \{ \dots | \tau'_i \} \dots \{ \dots | \tau'_j \} \{ \dots | \tau'_k \} \dots \{ \dots | \tau'_l \}$$

which completes the proof.

The right-hand sides of the group sums are also independent of the choice of sign for the 3j-symbol. Clearly, the left-hand side is also independent of this choice since each triad appears twice. The choice of sign is the same for a given triad (i.e. for a given set of j's, no matter what the m's are) and therefore in the group sums each sign appears twice, giving a plus.

It is therefore satisfactory to write these functions of 6j-symbols with the group sum relations and ask whether this definition is equivalent to that using the 3j-symbols. This question cannot be answered fully at this stage. We would have to examine the possibility of group sums made up of 5, 6, ... 6j-symbols and see whether these are independent of the four we have. If they were



not independent, one could then use the four group sums we have above and define the  $6j$ -symbols in terms of characters. As yet, we do not have a proof that there is no independent group sum with more than four  $6j$ -symbols in the product. However, we can suggest a way to attack this problem.

In each  $6j$ -symbol there are four triads and in a group sum formula, if we want invariance under an outer transformation, each triad must appear twice. How this is done in the four group sums, given in this chapter, can be seen with the use of diagrams. In these diagrams, each point corresponds to a  $6j$ -symbol in the product and each line corresponds to a triad. A line identifies a triad and joins the two  $6j$ -symbols (points) in which this triad appears. Thus, for theorem 4.7 there is only one point (one  $6j$ -symbol) and two pairs of triads so that we obtain the diagram of figure (4.5a). For theorem 4.6 we obtain figure (4-5b). For theorems 4.8 and 4.9 we obtain the diagrams in figures (4-5c) and (4-5d). It is easily seen that for these cases, i.e. for 1,2,3, and 4 points ( $6j$ -symbols), the diagrams are unique. In other words, for a given number of points, if we join the points by lines such that four lines meet at any point, the diagram obtained is topologically the same as the one shown in the figure. For five points this is no longer true, i.e. there are more than one independent diagram. In fact, there are five topologically different such diagrams and they are shown in figure (4-6). Of course, here as well as for 2,3, and 4 points we do not count the diagrams which can be reduced to simpler diagrams i.e. disconnected diagrams. In this case, the product of  $6j$ -symbols



can be broken down into simpler products.



FIGURE 4.5

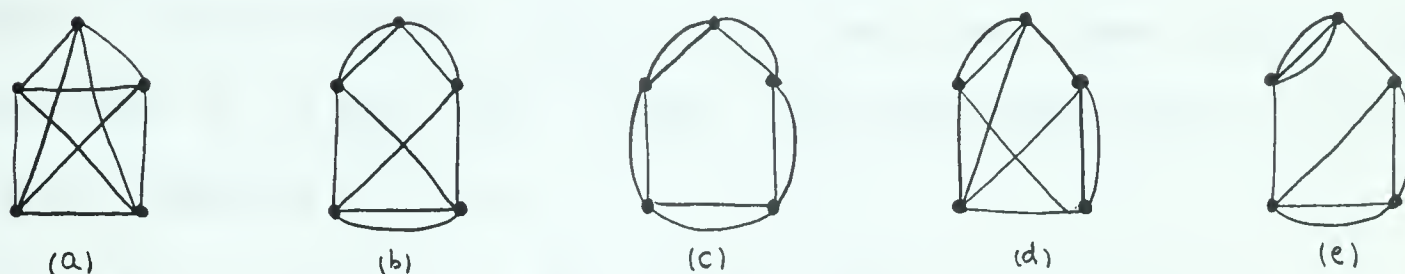


FIGURE 4.6

One would like to show that there is no group sum with five  $6j$ -symbols which is independent of the four group sums we know. To do this, one must prove that it is not possible to arrange the  $j$ 's in the five  $6j$ -symbols such that one of the diagrams of figure (4.6) is obtained. It would be interesting to see a more general proof, i.e. for a product of more than four  $6j$ -symbols we cannot form a new group sum.

The group sum properties of the  $6j$ -symbols are not very well known even for simply reducible groups. They can be useful in getting more insight for the properties of the  $6j$ -symbols.

We should note here that when each of the  $\tau$  variables takes on one value only, the group sums reduce to the expressions for multiplicity free groups (see Wigner or Sharp).



Numerical Example Using a Group Sum. We shall now give a numerical example showing how the group sum formulae can be used. Consider the group  $S_5$  (which is not multiplicity free). We consider theorem 4.7 with:

$$j_1 = (3, 1^2), \quad j_2 = (3, 2), \quad j' = (3, 1^2), \quad j = (2^2, 1).$$

Then  $\tau'$  runs over two values and  $\tau$  only has one value. For  $S_5$ , the order  $g$  is  $120 = 5!$ . Thus, the left-hand side of the group sum of theorem 4.7 is:

$$\begin{aligned} \sum_{\tau, \tau'} \left\{ \begin{array}{ccc|cc} j_1 & j_2 & j' & \tau' & \tau' \\ j_1 & j_2 & j & \tau & \tau \end{array} \right\} &= \sum_{\tau'=1}^2 \left\{ \begin{array}{ccc|cc} (3, 1^2) & (3, 2) & (3, 1^2) & \tau' & \tau' \\ (3, 1^2) & (3, 2) & (2^2, 1) & 1 & 1 \end{array} \right\} \\ &= \left\{ \begin{array}{ccc|cc} (3, 1^2) & (3, 2) & (3, 1^2) & 1 & 1 \\ (3, 1^2) & (3, 2) & (2^2, 1) & 1 & 1 \end{array} \right\} + \left\{ \begin{array}{ccc|cc} (3, 1^2) & (3, 2) & (3, 1^2) & 2 & 2 \\ (3, 1^2) & (3, 2) & (2^2, 1) & 1 & 1 \end{array} \right\}. \end{aligned}$$

On the other hand, the right-hand side can be computed from the class multiplication table given below.

Table 4.1

Class Multiplication Table For  $S_5$ .

Line No.	No. of cases	Class of R	Class of S	Class of SR	Class of $S^{-1}R$
1	1	$(1^5)$	$(1^5)$	$(1^5)$	$(1^5)$
2	10	$(1^5)$	$(21^3)$	$(21^3)$	$(21^3)$
3	15	$(1^5)$	$(221)$	$(221)$	$(221)$
4	20	$(1^5)$	$(31^2)$	$(31^2)$	$(31^2)$



Line No.	No. of Cases	Class of R	Class of S	Class of SR	Class of $S^{-1}R$
5	20	$(1^5)$	$(32)$	$(32)$	$(32)$
6	30	$(1^5)$	$(41)$	$(41)$	$(41)$
7	24	$(1^5)$	$(5)$	$(5)$	$(5)$
8	10	$(21^3)$	$(1^5)$	$(21^3)$	$(21^3)$
9	10	$(21^3)$	$(21^3)$	$(1^5)$	$(1^5)$
10	30	$(21^3)$	$(21^3)$	$(221)$	$(221)$
11	60	$(21^3)$	$(21^3)$	$(31^2)$	$(31^2)$
12	30	$(21^3)$	$(221)$	$(21^3)$	$(21^3)$
13	60	$(21^3)$	$(221)$	$(32)$	$(32)$
14	60	$(21^3)$	$(221)$	$(41)$	$(41)$
15	20	$(21^3)$	$(31^2)$	$(32)$	$(32)$
16	120	$(21^3)$	$(31^2)$	$(41)$	$(41)$
17	60	$(21^3)$	$(31^2)$	$(21^3)$	$(21^3)$
18	20	$(21^3)$	$(32)$	$(31^2)$	$(31^2)$
19	120	$(21^3)$	$(32)$	$(5)$	$(5)$
20	60	$(21^3)$	$(32)$	$(221)$	$(221)$
21	120	$(21^3)$	$(41)$	$(5)$	$(5)$
22	180	$(21^3)$	$(41)$	$(31^2)$	$(31^2)$
23	120	$(21^3)$	$(5)$	$(41)$	$(41)$
24	120	$(21^3)$	$(5)$	$(32)$	$(32)$
25	15	$(221)$	$(1^5)$	$(221)$	$(221)$
26	30	$(221)$	$(21^3)$	$(21^3)$	$(21^3)$
27	60	$(221)$	$(21^3)$	$(32)$	$(32)$
28	60	$(221)$	$(21^3)$	$(41)$	$(41)$



Line No.	No. of Cases	Class of R	Class of S	Class of SR	Class of $S^{-1}R$
29	15	(221)	(221)	(1 <sup>5</sup> )	(1 <sup>5</sup> )
30	30	(221)	(221)	(221)	(221)
31	60	(221)	(221)	(31 <sup>2</sup> )	(31 <sup>2</sup> )
32	120	(221)	(221)	(5)	(5)
33	120	(221)	(31 <sup>2</sup> )	(31 <sup>2</sup> )	(31 <sup>2</sup> )
34	120	(221)	(31 <sup>2</sup> )	(5)	(5)
35	60	(221)	(31 <sup>2</sup> )	(221)	(221)
36	60	(221)	(32)	(21 <sup>3</sup> )	(21 <sup>3</sup> )
37	120	(221)	(32)	(32)	(32)
38	120	(221)	(32)	(41)	(41)
39	90	(221)	(41)	(21 <sup>3</sup> )	(21 <sup>3</sup> )
40	240	(221)	(41)	(41)	(41)
41	120	(221)	(41)	(32)	(32)
42	120	(221)	(5)	(31 <sup>2</sup> )	(31 <sup>2</sup> )
43	120	(221)	(5)	(221)	(221)
44	120	(221)	(5)	(5)	(5)
45	20	(31 <sup>2</sup> )	(21 <sup>3</sup> )	(32)	(32)
46	60	(31 <sup>2</sup> )	(21 <sup>3</sup> )	(21 <sup>3</sup> )	(21 <sup>3</sup> )
47	120	(31 <sup>2</sup> )	(21 <sup>3</sup> )	(41)	(41)
48	60	(31 <sup>2</sup> )	(221)	(221)	(221)
49	120	(31 <sup>2</sup> )	(221)	(31 <sup>2</sup> )	(31 <sup>2</sup> )
50	120	(31 <sup>2</sup> )	(221)	(5)	(5)
51	20	(31 <sup>2</sup> )	(31 <sup>2</sup> )	(31)	(1 <sup>5</sup> )
52	20	(31 <sup>2</sup> )	(31 <sup>2</sup> )	(1 <sup>5</sup> )	(31)
53	120	(31 <sup>2</sup> )	(31 <sup>2</sup> )	(5)	(5)



Line No.	No. of Cases	Class of R	Class of S	Class of SR	Class of $S^{-1}R$
78	20	(32)	$(31^2)$	$(21^3)$	(32)
79	20	(32)	$(31^2)$	(32)	$(21^3)$
80	120	(32)	$(31^2)$	(41)	(41)
81	120	(32)	$(31^2)$	(32)	(41)
82	120	(32)	$(31^2)$	(41)	(32)
83	20	(32)	(32)	$(1^5)$	$(31^2)$
84	20	(32)	(32)	$(31^2)$	$(1^5)$
85	240	(32)	(32)	(221)	(221)
86	120	(32)	(32)	$(31^2)$	$(31^2)$
87	120	(32)	(41)	(221)	$(31^2)$
88	120	(32)	(41)	$(31^2)$	(221)
89	80	(32)	(41)	$(31^2)$	$(31^2)$
90	80	(32)	(41)	(5)	$(31^2)$
91	80	(32)	(41)	$(31^2)$	(5)
92	120	(32)	(41)	(5)	(5)
93	120	(32)	(5)	(41)	$(21^3)$
94	120	(32)	(5)	$(21^3)$	(41)
95	120	(32)	(5)	(32)	(41)
96	120	(32)	(5)	(41)	(32)
97	30	(41)	$(1^5)$	(41)	(41)
98	120	(41)	$(21^3)$	(5)	(5)
99	180	(41)	$(21^3)$	$(31^2)$	$(31^2)$
100	90	(41)	(221)	$(21^3)$	$(21^3)$
101	240	(41)	(221)	(41)	(41)
102	120	(41)	(221)	(32)	(32)



Line No.	No. of Cases	Class of R	Class of S	Class of SR	Class of $S^{-1}R$
103	120	(41)	$(31^2)$	(41)	$(21^3)$
104	120	(41)	$(31^2)$	$(21^3)$	(41)
105	180	(41)	$(31^2)$	(32)	(41)
106	180	(41)	$(31^2)$	(41)	(32)
107	120	(41)	(32)	(221)	$(31^2)$
108	120	(41)	(32)	$(31^2)$	(221)
109	80	(41)	(32)	$(31^2)$	$(31^2)$
110	80	(41)	(32)	(5)	$(31^2)$
111	80	(41)	(32)	$(31^2)$	(5)
112	120	(41)	(32)	(5)	(5)
113	30	(41)	(41)	(221)	$(1^5)$
114	30	(41)	(41)	$(1^5)$	(221)
115	120	(41)	(41)	$(31^2)$	$(31^2)$
116	120	(41)	(41)	(5)	$(31^2)$
117	120	(41)	(41)	$(31^2)$	(5)
118	240	(41)	(41)	(5)	(221)
119	240	(41)	(41)	(221)	(5)
120	240	(41)	(5)	(41)	(41)
121	120	(41)	(5)	(32)	$(21^3)$
122	120	(41)	(5)	$(21^3)$	(32)
123	120	(41)	(5)	(41)	(32)
124	120	(41)	(5)	(32)	(41)
125	24	(5)	$(1^5)$	(5)	(5)
126	120	(5)	$(21^3)$	(41)	(41)



Line No.	No. of Cases	Class of R	Class of S	Class of SR	Class of $S^{-1}R$
127	120	(5)	$(21^3)$	(32)	(32)
128	120	(5)	(221)	$(31^2)$	$(31^2)$
129	120	(5)	(221)	(221)	(221)
130	120	(5)	(221)	(5)	(5)
131	120	(5)	$(31^2)$	(5)	(221)
132	120	(5)	$(31^2)$	(221)	(5)
133	120	(5)	$(31^2)$	(5)	$(31^2)$
134	120	(5)	$(31^2)$	$(31^2)$	(5)
135	120	(5)	(32)	(41)	$(21^3)$
136	120	(5)	(32)	$(21^3)$	(41)
137	120	(5)	(32)	(32)	(41)
138	120	(5)	(32)	(41)	(32)
139	240	(5)	(41)	(41)	(41)
140	120	(5)	(41)	(32)	$(21^3)$
141	120	(5)	(41)	$(21^3)$	(32)
142	120	(5)	(41)	(41)	(32)
143	120	(5)	(41)	(32)	(41)
144	24	(5)	(5)	$(1^5)$	(5)
145	24	(5)	(5)	(5)	$(1^5)$
146	48	(5)	(5)	(5)	(5)
147	120	(5)	(5)	$(31^2)$	(221)
148	120	(5)	(5)	(221)	$(31^2)$
149	120	(5)	(5)	(5)	$(31^2)$
150	120	(5)	(5)	$(31^2)$	(5)



The standard table of character for  $S_5$  is shown below.

Table 4.2

Character Table For  $S_5$ .

Irreducible Representation \ No. of elements	1 (1 <sup>5</sup> )	10 (1 <sup>3</sup> , 2)	20 (1 <sup>2</sup> , 3)	15 (1, 2 <sup>2</sup> )	30 (1, 4)	20 (2, 3)	24 (5)
(5)	1	1	1	1	1	1	1
(4, 1)	4	2	1	0	0	-1	-1
(3, 2)	5	1	-1	1	-1	1	0
(3, 1 <sup>2</sup> )	6	0	0	-2	0	0	1
(2 <sup>2</sup> , 1)	5	-1	-1	1	1	-1	0
(2, 1 <sup>3</sup> )	4	-2	1	0	0	1	-1
(1 <sup>5</sup> )	1	-1	1	1	-1	-1	1

The right-hand side of theorem 4.7 is:

$$(1/120)^2 \sum_{R, S} \chi^{(3,1^2)}(R) \chi^{(2^3,1)}(S) \chi^{(3,1^2)}(S^{-1}R) \chi^{(3,2)}(SR).$$

Only a few terms under the summation will contribute; i.e. those for which none of the four factors vanishes. These terms are shown in table 4.3.



Table 4.3

Line No.	Weight $W$	$\chi^{(3,1)}(R)$	$\chi^{(2,1)}(S)$	$\chi^{(3,1)}(SR)$	$\chi^{(3,2)}(S^{-1}R)$	Product $\pi$	$W\pi$
1	1	6	5	6	5	900	900
3	15	6	1	-2	1	-12	-180
25	15	-2	5	-2	1	20	300
29	15	-2	1	6	5	-60	-900
30	30	-2	1	-2	1	4	120
35	60	-2	-1	-2	1	-4	-240
129	120	1	1	-2	1	-2	-240
131	120	1	-1	1	1	-1	-120
133	120	1	-1	1	-1	1	120

Thus, the right-hand side of theorem 4.7 for this case is:

$$[1/(120)^2](-240) = -1/60.$$

It follows that:

$$\left\{ \begin{array}{ccc|cc} (3,1) & (3,2) & (3,1) & 1 & 1 \\ (3,1) & (3,2) & (2,1) & 1 & 1 \end{array} \right\} + \left\{ \begin{array}{ccc|cc} (3,1) & (3,2) & (3,1) & 2 & 2 \\ (3,1) & (3,2) & (2,1) & 1 & 1 \end{array} \right\} = -1/60.$$

We have thus obtained a numerical value for the sum of two 6j-symbols. These differ only in the  $\tau$  variable. The actual value of each of the two 6j-symbols depends on the choice of basis since each one is determined only up to an outer basis transformation. By a suitable choice of basis, any one of the two



could be made to vanish. However, as mentioned in the previous section the sum of the two is invariant.

6j-Symbol Containing the Identity Representation. We shall now consider another special case. Let the group G be ambivalent and let it have only integer representations and consider the following 6j-symbol:

$$\left\{ \begin{array}{ccc} j_1 & j_1' & 0 \\ j_2 & j_2' & j \end{array} \middle| \begin{array}{c} 11 \\ \tau \tau' \end{array} \right\} ,$$

where 0 represents the identity representation. The identity representation is contained at most once in the product of any two irreducible representations. By definition:

$$\left\{ \begin{array}{ccc} j_1 & j_1' & 0 \\ j_2 & j_2' & j \end{array} \middle| \begin{array}{c} 11 \\ \tau \tau' \end{array} \right\} = (j_1 j_1' 0 1)_{m_1 m_1'} (j_1 j_2' j \tau)_{m_1 m_2' m} \\ \times (j_2 j_1' j \tau')_{m_2 m_1' m} (j_2 j_2' 0 1)_{m_2 m_2' 1}$$

Using equ.(2-44) we have that:

$$(j_1 j_2 0 1)_{m_1 m_2 1} = 1/\sqrt{n_{j_1}} \delta_{j_1 j_2} \delta_{m_1 m_2} ,$$

Hence:

$$\left\{ \begin{array}{ccc} j_1 & j_1' & 0 \\ j_2 & j_2' & j \end{array} \middle| \begin{array}{c} 11 \\ \tau \tau' \end{array} \right\} = \delta_{j_1 j_2} \delta_{j_1 j_2'} \delta_{m_1 m_1'} \delta_{m_2 m_2'} \frac{1}{\sqrt{n_{j_1} n_{j_2}}} (j_1 j_2' j \tau)_{m_1 m_2' m} (j_2 j_1' j \tau')_{m_2 m_1' m} \\ = \delta_{j_1 j_2} \delta_{j_2 j_2'} \frac{1}{\sqrt{n_{j_1} n_{j_2}}} (j_1 j_2 j \tau)_{m_1 m_2 m} (j_2 j_1 j \tau')_{m_2 m_1 m}$$



and using the symmetry properties of the 3j-symbols we obtain:

$$\left\{ \begin{matrix} j_1, j_1' & 0 \\ j_2, j_2' & j \end{matrix} \middle| \begin{matrix} 1' \\ \tau, \tau' \end{matrix} \right\} = \delta_{j_1, j_1'} \delta_{j_2, j_2'} \frac{f_c(j_1, j_2, j, \tau)}{\sqrt{n_{j_1, n_{j_2}}}} (j_1, j_2, j, \tau)_{m_1, m_2, m} (j_1, j_2, j, \tau')_{m_1, m_2, m}$$

Using the orthogonality of the 3j-symbols this becomes:

$$\begin{aligned} \left\{ \begin{matrix} j_1, j_1' & 0 \\ j_2, j_2' & j \end{matrix} \middle| \begin{matrix} 1' \\ \tau, \tau' \end{matrix} \right\} &= \delta_{j_1, j_1'} \delta_{j_2, j_2'} \frac{1}{\sqrt{n_{j_1, n_{j_2}}}} \frac{1}{n_j} \delta_{\tau\tau'} \delta_{mm} f_c(j_1, j_2, j, \tau) \\ &= \frac{1}{\sqrt{n_{j_1, n_{j_2}}}} \delta_{j_1, j_1'} \delta_{j_2, j_2'} \delta_{\tau\tau'} f_c(j_1, j_2, j, \tau). \end{aligned}$$

Hence:

$$\left\{ \begin{matrix} j_1, j_1' & 0 \\ j_2, j_2' & j \end{matrix} \middle| \begin{matrix} 1' \\ \tau, \tau' \end{matrix} \right\} = f_c(j_1, j_2, j, \tau) \frac{1}{\sqrt{n_{j_1, n_{j_2}}}} \delta_{j_1, j_1'} \delta_{j_2, j_2'} \delta_{\tau\tau'}. \quad (4-22)$$

In this chapter, we have studied the properties of the 6j-symbols for ambivalent groups with integer representations. We have seen that the usual properties for the 6j-symbols for simply reducible groups have their equivalent properties for groups which are not multiplicity free. In many cases, sums over the  $\tau$  variables are involved. For the symmetry relations, we must be careful in dealing with a 6j-symbol which contains two equal representations. In some of these cases, exchanging triads in the 6j-symbol can change its sign (at most). For the symmetric group there is an additional symmetry from conjugating the representations of two columns in the 6j-symbol. Here, one must be careful for selfconjugate representations which in some cases can



introduce a change of sign. As before, these symmetries can introduce at most a sign change.

We have also seen that the  $6j$ -symbols are real and that they are invariant under an inner basis transformation. However, they are only defined up to an outer basis transformation.



## CHAPTER 5

### Conclusion and Outlook.

In physical application, we are often concerned with the evaluation of tensor operators between two states. The concept of an irreducible tensor operator was introduced by Racah<sup>1</sup> for the rotation group. Later, Racah<sup>7</sup> generalized this concept to that of an irreducible tensor operator for any group. In 1958, Koster<sup>8</sup> generalized the Wigner-Eckart theorem to groups other than the rotation group.

For a group  $G$  a set of operators  $Q_i^\mu$  ( $i = 1, 2, 3, \dots$ ) are said to be irreducible tensor operators if they transform like the basis vectors of the irreducible representation  $\mu$  of  $G$ , i.e.:

$$D(R) Q_i^\mu D^{-1}(R) = \sum_j Q_j^\mu D_{ji}^\mu(R) \quad (5-1)$$

The Wigner-Eckart theorem for groups which are not necessarily multiplicity free becomes:

$$\begin{aligned} \langle \alpha_1 j_1 m_1 | Q_i^\mu | \alpha_2 j_2 m_2 \rangle \\ = \sum_{\tau=1}^{c(\alpha_1 j_1)} \langle \alpha_1 j_1 || Q^\mu || \alpha_2 j_2 \rangle_\tau S_{m_1 i m_2}^{\tau \mu} \end{aligned} \quad (5-2)$$

where the "reduced matrix elements"  $\langle \alpha_1 j_1 || Q^\mu || \alpha_2 j_2 \rangle_\tau$  are independent of  $m_1$ ,  $m_2$ , and  $i$ . The letter  $\alpha$  stands for all quantum numbers other than  $j$  and  $m$  which may be required to describe the state. The proof of equ.(5-2) can be found in the appendix on group theory



of the book of Messiah "Mécanique quantique"<sup>9</sup>. The difference between equ.(5-2) and the ordinary Wigner-Eckart theorem is in the sum over  $\tau$ . In equ.(5-2), we see that there are  $C(\mu j_2 j_1)$  reduced matrix elements for each set of  $\mu$ ,  $j_2$ , and  $j_1$ . Clearly, for groups which are not multiplicity free the calculation of matrix elements can be much more involved.

L. Cohen<sup>10</sup> has already made use of the  $3j$ -symbols (which he calls coupling coefficients) for the group  $S_6$ . We have seen that  $S_6$  is not multiplicity free. He has constructed wave functions for  $Li_6$  in a fairly general way. In the discussion, he mentions that he has evaluated some matrix elements. He ends the paper by writing: "The most lengthy part of the calculation is that which involves summation over products of the coupling coefficients of the symmetric group." It seems natural to believe that what Cohen refers to is simply our  $6j$ -symbols. We think that it would be preposterous to believe that the  $6j$ -symbols for  $S_n$  will become as important in theoretical physics as those for  $R_3$ . However, there are indications which suggest that they are of interest for applications to nuclear physics. We have mentioned the paper of Cohen, but also the fact that equ.(5-2) can be found as a theorem in a standard textbook<sup>9</sup> indicates the interest for groups which are not multiplicity free.

Another less obvious result of this study is that it can give us more insight into the fundamental nature of some of the properties of the  $6j$ -symbols for simply reducible groups.



# APPENDIX

When two numbers follow an equation, the first one is the number assigned in the book of Hamermesh. The unprimed numbers refer to equations as found in Hamermesh. The primed numbers indicate the notation of Sharp is used. Thus, (A1) and (A1)' refer to the same equation written in the notations of Hamermesh and Sharp respectively.

$$D_{ij}^M(R) D_{kl}^V(R) S_{s \quad j \quad l}^{\lambda \tau \lambda \quad \mu \nu} = D_{s's'}^{\lambda \tau \lambda}(R) S_{s' \quad i \quad k}^{\lambda \tau \lambda \quad \mu \nu} \quad (7-186), (A1)$$

$$D_{ij}^M(R) D_{kl}^V(R) (\mu \nu \lambda \tau \lambda)_{j \quad k \quad l} = D_{s's'}^{\lambda \tau \lambda}(R) (\mu \nu \lambda \tau \lambda)_{i \quad k \quad s'} \quad (A1)'$$

$$S_{i \quad j \quad k}^{\lambda \tau \lambda \quad \mu \nu} S_{i' \quad j' \quad k'}^{\epsilon \tau \epsilon \quad \mu \nu} = \delta_{\lambda \epsilon} \delta_{\tau \lambda \tau \epsilon} \delta_{i i'} \quad (7-188), (A2)$$

$$n_{\lambda} (\mu \nu \lambda \tau \lambda)_{j \quad k \quad i} (\mu \nu \epsilon \tau \epsilon)_{j' \quad k' \quad i'} = \delta_{\lambda \epsilon} \delta_{\tau \lambda \tau \epsilon} \delta_{i i'} \quad (A2)'$$

$$\sum_{\lambda \tau \lambda} S_{i \quad j \quad k}^{\lambda \tau \lambda \quad \mu \nu} S_{i' \quad j' \quad k'}^{\lambda \tau \lambda \quad \mu \nu} = \delta_{j j'} \delta_{k k'} \quad (7-188a), (A2a)$$

$$\sum_{\lambda \tau \lambda} n_{\lambda} (\mu \nu \lambda \tau \lambda)_{j \quad k \quad i} (\mu \nu \lambda \tau \lambda)_{j' \quad k' \quad i'} = \delta_{j j'} \delta_{k k'} \quad (A2a)'$$

$$S_{t \quad i \quad k}^{\lambda' \tau \lambda' \quad \mu \nu} D_{ij}^M(R) D_{kl}^V(R) S_{s \quad j \quad l}^{\lambda \tau \lambda \quad \mu \nu} = D_{s's'}^{\lambda \tau \lambda}(R) \delta_{\lambda \lambda'} \delta_{\tau \lambda \tau \lambda'} \delta_{t s'} \quad (7-189), (A3)$$

$$n_{\lambda} (\mu \nu \lambda' \tau \lambda')_{i \quad k \quad t} D_{ij}^M(R) D_{kl}^V(R) (\mu \nu \lambda \tau \lambda)_{j \quad k \quad l} = D_{s's'}^{\lambda \tau \lambda}(R) \delta_{\lambda \lambda'} \delta_{\tau \lambda \tau \lambda'} \delta_{t s'} \quad (A3)'$$

$$D_{ij}^M(R) D_{kl}^V(R) = \sum_{\lambda, \tau \lambda} S_{s' \quad i \quad k}^{\lambda \tau \lambda \quad \mu \nu} D_{s's'}^{\lambda \tau \lambda}(R) S_{s \quad j \quad l}^{\lambda \tau \lambda \quad \mu \nu} \quad (7-190), (A4)$$

$$D_{ij}^M(R) D_{kl}^V(R) = \sum_{\lambda, \tau \lambda} n_{\lambda} (\mu \nu \lambda \tau \lambda)_{i \quad k \quad s'} D_{s's'}^{\lambda \tau \lambda}(R) (\mu \nu \lambda \tau \lambda)_{j \quad k \quad s} \quad (A4)'$$



$$D_{ts}^{\lambda}(R) D_{ij}^{\mu}(R) D_{kl}^{\nu}(R) S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \ \nu} = S_{t \ i \ k}^{\lambda \tau \lambda \ \mu \ \nu} \quad (7-191), (A5)$$

$$D_{ts}^{\lambda}(R) D_{ij}^{\mu}(R) D_{kl}^{\nu}(R) (\mu \nu \lambda \tau \lambda)_{j \ l \ s} = (\mu \nu \lambda \tau \lambda)_{i \ k \ t} \quad (A5)'$$

$$\sum_R D_{ts}^{\lambda}(R) D_{ij}^{\mu}(R) D_{kl}^{\nu}(R) = g/n_{\lambda} \sum_{\tau_{\lambda}} S_{t \ i \ k}^{\lambda \tau \lambda \ \mu \ \nu} S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \ \nu} \quad (7-192)^*, (A6)$$

$$\sum_R D_{ts}^{\lambda}(R) D_{ij}^{\mu}(R) D_{kl}^{\nu}(R) = g \sum_{\tau_{\lambda}} (\mu \nu \lambda \tau \lambda)_{i \ k \ t} (\mu \nu \lambda \tau \lambda)_{j \ l \ s} \quad (A6)'$$

$$(1/n_{\lambda}) \sum_{\tau_{\lambda}} [S_{t \ i \ k}^{\lambda \tau \lambda \ \mu \ \nu}]^2 = (1/n_{\mu}) \sum_{\tau_{\mu}} [S_{i \ t \ k}^{\mu \tau \mu \ \lambda \ \nu}]^2 = \text{etc.} \quad (7-193)^*, (A7)$$

$$\sum_{\tau_{\lambda}} (\mu \nu \lambda \tau \lambda)_{i \ k \ t}^2 = \sum_{\tau_{\mu}} (\lambda \nu \mu \tau \mu)_{t \ k \ i}^2 = \text{etc.} \quad (A7)'$$

$$\sqrt{n_{\mu}} S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \ \nu} = \sqrt{n_{\lambda}} S_{j \ s \ l}^{\mu \tau \lambda \ \lambda \ \nu} \quad (7-205), (A8)$$

$$(\mu \nu \lambda \tau \lambda)_{j \ l \ s} = (\lambda \nu \mu \tau \lambda)_{s \ l \ j} \quad (A8)'$$

$$\sqrt{n_{\nu}} S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \ \nu} = \sqrt{n_{\lambda}} S_{l \ j \ s}^{\nu \tau \lambda \ \mu \ \lambda} \quad (7-205a), (A8a)$$

$$(\mu \nu \lambda \tau \lambda)_{j \ l \ s} = (\mu \lambda \nu \tau \lambda)_{j \ s \ l} \quad (A8a)'$$

$$\left. \begin{aligned} S_{s \ j \ l}^{\lambda \tau \lambda \ \mu \ \nu} &= S_{s \ l \ j}^{\lambda \tau \lambda \ \nu \ \mu} \\ S_{j \ s \ l}^{\mu \tau \lambda \ \lambda \ \nu} &= S_{j \ l \ s}^{\mu \tau \lambda \ \nu \ \lambda} \\ S_{l \ j \ s}^{\nu \tau \lambda \ \mu \ \lambda} &= S_{l \ s \ j}^{\nu \tau \lambda \ \lambda \ \mu} \end{aligned} \right\} \quad (7-205b), (A8b)$$

\* There seems to be a misprint in the book of Hamermesh.



$$\left. \begin{aligned} (\mu \nu \lambda \tau \lambda)_{j \ell s} &= (\nu \mu \lambda \tau \lambda)_{\ell j s} \\ (\lambda \nu \mu \tau \lambda)_{s \ell j} &= (\nu \lambda \mu \tau \lambda)_{\ell s j} \\ (\mu \lambda \nu \tau \lambda)_{j s \ell} &= (\lambda \mu \nu \tau \lambda)_{s j \ell} \end{aligned} \right\} \quad (A8b)'$$

In equations (A8), (A8a), and (A8b) it is assumed that  $\mu \neq \nu \neq \lambda \neq \mu$ .

When two irreducible representations are equal, we have:

$$\begin{aligned} \frac{1}{\sqrt{n_\lambda}} S_{s \ j \ \lambda}^{\lambda \tau \lambda \ \mu \ \mu} &= \frac{1}{\sqrt{n_\lambda}} \delta_{\tau \lambda} S_{s \ \ell \ j}^{\lambda \tau \lambda \ \mu \ \mu} = \frac{1}{\sqrt{n_\mu}} S_{j \ s \ \ell}^{\mu \tau \lambda \ \lambda \ \mu} = \frac{1}{\sqrt{n_\mu}} S_{j \ \ell \ s}^{\mu \tau \lambda \ \mu \ \lambda} \\ &= \frac{1}{\sqrt{n_\mu}} \delta_{\tau \lambda} S_{\ell \ s \ j}^{\mu \tau \lambda \ \lambda \ \mu} = \frac{1}{\sqrt{n_\mu}} \delta_{\tau \lambda} S_{\ell \ j \ s}^{\mu \tau \lambda \ \mu \ \lambda} \quad (7-206c), (A9) \end{aligned}$$

$$\begin{aligned} (\mu \mu \lambda \tau \lambda)_{j \ell s} &= \delta_{\tau \lambda} (\mu \mu \lambda \tau \lambda)_{\ell j s} = (\lambda \mu \mu \tau \lambda)_{s \ell j} = (\mu \lambda \mu \tau \lambda)_{\ell s j} \\ &= \delta_{\tau \lambda} (\lambda \mu \mu \tau \lambda)_{s j \ell} = \delta_{\tau \lambda} (\mu \lambda \mu \tau \lambda)_{j s \ell} \quad (A9)' \end{aligned}$$

Finally, when the three irreducible representations are the same, we have:

$$\begin{aligned} S_{s \ j \ \lambda}^{\lambda \tau \lambda \ \lambda \ \lambda} &= \delta_{\tau \lambda} S_{s \ \ell \ j}^{\lambda \tau \lambda \ \lambda \ \lambda} = \epsilon_{\tau \lambda} S_{j \ s \ \ell}^{\lambda \tau \lambda \ \lambda \ \lambda} \\ &= \epsilon_{\tau \lambda} \delta_{\tau \lambda} S_{j \ \ell \ s}^{\lambda \tau \lambda \ \lambda \ \lambda} = \delta_{\tau \lambda} S_{\ell \ j \ s}^{\lambda \tau \lambda \ \lambda \ \lambda} = \epsilon_{\tau \lambda} \delta_{\tau \lambda} S_{\ell \ s \ j}^{\lambda \tau \lambda \ \lambda \ \lambda} \quad (7-207b), (A10) \end{aligned}$$

$$\begin{aligned} (\lambda \lambda \lambda \tau \lambda)_{j \ell s} &= \delta_{\tau \lambda} (\lambda \lambda \lambda \tau \lambda)_{\ell j s} = \epsilon_{\tau \lambda} (\lambda \lambda \lambda \tau \lambda)_{s \ell j} \\ &= \epsilon_{\tau \lambda} \delta_{\tau \lambda} (\lambda \lambda \lambda \tau \lambda)_{\ell s j} = \delta_{\tau \lambda} (\lambda \lambda \lambda \tau \lambda)_{j s \ell} = \epsilon_{\tau \lambda} \delta_{\tau \lambda} (\lambda \lambda \lambda \tau \lambda)_{s j \ell} \quad (A10)' \end{aligned}$$



When the fact that  $\epsilon_{\tau\lambda} = \delta_{\tau\lambda}$  is used, (A10)' can be written:

$$\begin{aligned} (\lambda\lambda\lambda\tau\lambda)_{j\lambda s} &= \delta_{\tau\lambda} (\lambda\lambda\lambda\tau\lambda)_{\lambda j s} = \delta_{\tau\lambda} (\lambda\lambda\lambda\tau\lambda)_{s\lambda j} \\ &= (\lambda\lambda\lambda\tau\lambda)_{\lambda s j} = \delta_{\tau\lambda} (\lambda\lambda\lambda\tau\lambda)_{j s \lambda} = (\lambda\lambda\lambda\tau\lambda)_{s j \lambda} . \quad (A10a)' \end{aligned}$$



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